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Professor R. A. Sampson, M.A., D.Sc., F.R.S., PRESIDENT, in the Chair.

John Botterill Rodney Haines, F. R. Met. Soc., 14 Chaucer Road, Wanstead, Essex; and H.M.S. *Umgeni*, c/o G.P.O., London;

Gertrude Longbottom, M.A., The Hollies, Louth, Lincolnshire; and

Mrs. Proctor-Smyth, 9 Orchard Road, Altrincham, Cheshire,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

James Hartness, Springfield, Vermont, U.S.A. (proposed by Ambrose Swasey);

Mrs. Annie Scott Dill Maunder, 8 Maze Hill, Greenwich Park, London, S.E. (proposed by E. Walter Maunder); and

Reginald Lawson Waterfield, The Principal's House, Cheltenham (proposed by W. H. Steavenson).

Fifty-six presents were announced as having been received since the last meeting, including, amongst others:—

An autograph dedication of the Georgium Sidus, by Sir William Herschel, presented by Miss Herschel.

On the Theory of Star-Streaming, and the Structure of the Universe.
(Second Paper). By J. H. Jeans, M.A., F.R.S.

1. The present paper is based upon an extension of the set of ideas which were put forward in an earlier paper of the same title.* Reasons were there given for thinking that the ascertained laws of star-streaming could not legitimately be interpreted in terms of the hypothesis that the universe had already reached a steady state. The occurrence of star-streaming such as is observed was itself shown to provide weighty evidence against this hypothesis, and it was suggested that the observed phenomena could perhaps be best explained in the simplest and most obvious way, as arising from two or more clusters of stars moving through one another to take up their final motion or positions in the universe.

A cluster of stars will suffer continual disintegration from its encounters with other stars or clusters. In a previous paper,† I investigated the rate at which this disintegration might be expected to occur, and found it to be comparatively very slow. If a universe lasted for an unlimited time, every cluster would have become wholly disintegrated, and the universe might be thought of as a chaos of separate and independent stars. But in the earlier stages of its evolution it is conceivable that a universe is better thought of, at least in part, as a chaos of moving star-clusters. In later stages of development, the moving star-clusters, although partially disintegrated, may still be recognisable as star-streams. Finally, the clusters will have become so scattered as to be no longer recognisable.

At first the streams may be so compact that it will be rare for two of them to intermingle. In a second stage, the extent in space of the streams may have widened so much that two or three streams may be expected to have intermingled in every region of space, there being mixed with them a number of independent stars which have become entirely separated from their streams. In a third stage, there will be a large number of intermingling streams everywhere, and the proportion of runaway or independent stars will have greatly increased. In the final stage, in which the streams are no longer recognisable as such, the chaos of moving clusters with which we started will have given place to a chaos of independently moving stars.

It would be absurd to imagine that the complex phenomena of our universe could be entirely explained in terms of simple conceptions such as these, but it is perhaps worth investigating whether some of its features may not admit of explanation in this way. And this leads us to consider the development of a fictitious universe, which we shall suppose to originate entirely

* *M. N.*, lxxiv. p. 109.

† *M. N.*, lxxiv. p. 109. See also Eddington, *Stellar Movements*, p. 253.

from moving clusters or star-streams. If our conjectures bear any approximation to the truth, our universe ought to be, in some ways at least, comparable with the fictitious universe in the second of the stages just sketched out, and some of the phenomena to be expected in the earlier stages ought perhaps to be still recognisable in our own universe.

2. Consider as hypothetical raw material for a universe a number of compact moving star-clusters, such that all the stars in any one cluster have approximately the same velocity and are not far removed from one another in space. Such a universe may be thought of as a rare gas, in which the molecules are star-clusters. "Collisions" are infrequent; when they take place the "molecules" pass through one another.* Any of the known globular clusters may be thought of as examples of "molecules" not in collision; perhaps the well-known cluster in Taurus may be thought of as providing an example of a molecule in collision.

The forces acting on a single star of such a cluster may be divided into three parts:

(a) The forces acting from the remaining stars of the cluster to which the star belongs.

(b) The forces acting from the stars in clusters which are in collision with this cluster.

(c) The forces acting from the stars in all other clusters.

When the development of the universe has proceeded to its second stage, the forces on any star may still be classified in the same way, but the distinction between the three types of forces will no longer be a perfectly clearly defined one. The classification will then give a convenient approximation to the truth, but will not express the exact truth.

For a fairly compact cluster the effect of forces of type (c) will be approximately the same for each star of the cluster. Thus the relative motion of the stars in a cluster will hardly be altered by these forces; their effect is to alter, accelerate, or retard the motion of the cluster as a whole, but not to alter its "intramolecular" motion. Thus, to an approximation which will be good in the earlier stages of development of our universe, we may suppose the disintegration of the cluster to be influenced by forces of types (a) and (b) only.

If one cluster existed alone in the universe, all the forces in action would be of type (a), and each star in the cluster would describe an orbit under the influence of these forces. If the forces were allowed to act for an infinite time, it can readily be shown that the cluster, under their influence alone, would spread out to infinite size, the law of density being the well-

* I have found it convenient to use the word "collision" by analogy with the Theory of Gases. A collision of clusters will not of course imply a material collision of the stars of which a cluster is formed, any more than a collision of molecules implies a material collision of the electrons of which a molecule is formed.

known "isothermal" law in which the density, except near the centre of the cluster, falls off inversely as the square of the distance from the centre. With this law, the cluster can only have finite mass by the density being infinitesimal everywhere except at the very centre. Thus it appears that the forces of type (a) would completely effect the disintegration of the cluster if they were allowed sufficient time. In the present paper, however, we shall be almost entirely concerned with the disintegrating effect of forces of type (b).

It is difficult to estimate the relative importance of the two sets of forces of types (a) and (b) in general terms. It is, however, easy to construct cases in which the forces of type (a) are entirely negligible in comparison with those of type (b). Such a case occurs when the stars in the primary cluster are very small in mass compared with those in the colliding cluster, or again, when the stars in the primary are much more loosely spaced than those in the colliding cluster. It is, of course, equally possible to construct special cases in which exactly opposite conditions prevail, so that the forces of type (b) will be insignificant in comparison with those of type (a). Our investigation will not be concerned with cases of this kind. For the present we shall be content to examine the effect of forces of type (b) only, bearing in mind that the neglect of forces of type (a) will certainly involve some error in every real problem, which may be large or small according to circumstances. (We shall attempt to estimate the amount of this error later.) Or, alternatively, we may suppose that we are investigating a problem in which there are ideal conditions, such as those suggested above, so that the forces of type (a) really are negligible in comparison with those of type (b).

3. With this understanding, we may proceed to investigate the dynamics of collision when one moving cluster encounters another moving cluster or a random assemblage of stars.

Let us examine the simplest problem first. Suppose that the two clusters meet in a head-on encounter; let the velocities, star-densities and masses in the two streams be denoted by v_1, ν_1, m_1 and v_2, ν_2, m_2 respectively; and let it be supposed for simplicity that the second cluster is large enough to envelop the first cluster entirely while the collision lasts.

The relative velocity of any two stars in opposite clusters is $v_1 + v_2$; the velocity of their centre of gravity, measured in the same direction as v_1 , is

$$v_G = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2}.$$

The velocity of the first star before the encounter, measured relative to the centre of gravity, is

$$v_1 - v_G = \frac{m_2}{m_1 + m_2} (v_1 + v_2).$$

Suppose the two stars now under consideration to pass one another at a distance σ sufficiently small to produce an appreciable, although not very large, deflection in their paths. The effect of the encounter is readily shown to be that the velocity of each star relative to the centre of gravity of the pair has its direction deflected through an angle ψ given by *

$$\psi = \frac{2(m_1 + m_2)}{(v_1 + v_2)^2 \sigma} \quad . \quad . \quad . \quad (1)$$

This formula is only accurate when ψ is small, but it is known that the deflections produced in a star's path may, for the majority of stars, be regarded as an accumulation of small deflections.† It follows that a good approximation is obtained by neglecting the errors in formula (1) in the few cases in which such errors are appreciable.

The velocity of the first star after encounter is now seen to consist of

(i) a component, approximately equal to v_1 , in the original direction of motion ;

(ii) a component, at right angles to the original direction of motion, of amount

$$(v_1 - v_G)\psi = \frac{2m_2}{(v_1 + v_2)\sigma} = \frac{2m_2}{V\sigma} \quad . \quad . \quad . \quad (2)$$

where V is the relative velocity.

If the two streams do not meet in a head-on encounter, it is easily seen that the motion of the first star can still be regarded as made up of its original motion together with a superposed velocity

$$\frac{2m_2}{V\sigma} \quad . \quad . \quad . \quad . \quad (3)$$

at right angles to the direction of relative velocity V . For this formula is unaltered on referring it to moving axes, and by a suitable choice of motion for the axes any encounter can be changed into a head-on encounter.

It is important to notice that formula (3) does not involve m_1 , the mass of the star whose motion is under consideration.

4. Consider now the encounters of ν_1 stars occupying a unit volume of the first stream. In an interval t , during which this unit volume is completely surrounded by the second stream, the number of encounters experienced by these ν_1 stars, such that σ is less than some assigned distance σ_0 , is

$$\nu_1 \nu_2 \pi \sigma_0^2 V t,$$

and of these the number for which σ lies within certain narrow limits σ and $\sigma + d\sigma$ will be

$$2\nu_1 \nu_2 \pi \sigma V t \, d\sigma = \nu_1 N, \text{ say.}$$

* *M. N.*, lxxiv. p. 111.

† *L.c.* p. 112.

These collisions are independent events in the sense that a star which has experienced one collision is just as likely to experience a second as it would have been if the first had not taken place. The first collision slightly bends the path of the star, but the influence of this on its chance of collision is only a small quantity of the second order.

It follows that the law of distribution of the $\nu_1 N$ collisions between the ν_1 stars is the same as that of $\nu_1 N$ articles distributed at random into ν_1 pigeon-holes. Thus each star will experience exactly N collisions of the kind under consideration, except for small deviations which are of the order of $N^{\frac{1}{2}}$, and so are insignificant in comparison with N .

For any star each of the N collisions for which σ lies within the narrow limits σ and $\sigma + d\sigma$ will superpose on to the velocity of the star a velocity of the amount given by formula (3). The direction of this superposed velocity is in the plane perpendicular to the direction of V , and all directions in this plane are equally likely. Compounding all the velocities superposed by the different collisions, we find that the effect on the first cluster of mingling with a second enveloping cluster is to produce in the former a symmetrical distribution of velocities in the plane perpendicular to V , these velocities being superposed on to the original velocities of the cluster in space; the stars scatter at right angles to the direction of the collision.

Thus the velocity diagram of the cluster is changed by the encounter we have considered from being (approximately) a point into being a disc. The distribution in space of the stars in the cluster will, as time progresses, be changed in the same way, so that from being (approximately) a point, the cluster will become a disc, which will continue to expand as time goes on. The two discs will, of course, be in the same plane, that normal to the relative velocity V .

5. This general result will at once recall Professor Turner's description of the Ursa Major cluster.* This cluster has the shape of a very flat disc perpendicular to the galactic plane, while the cluster itself is moving in the galactic plane. These data can be accounted for in a very simple manner by supposing that the extension of the cluster is due to its having met only one other cluster since its birth, this second cluster also having had a velocity in the galactic plane.

If there is anything in these ideas we may, in imagination at least, contemplate the possibility of unravelling the past history of a cluster from a consideration of its present velocity diagram and extension in space. The clusters which will be simplest to treat in this way will be disc-shaped clusters, for which the velocity diagram is a parallel disc. Such a cluster may be supposed to owe its shape to an encounter with a single other star-stream, or perhaps to a succession of encounters with

* *The Observatory*, 34, p. 246.

parallel moving star-streams. The former case will be distinguishable by the velocity diagram being a figure exactly similar to, as well as parallel to, the cluster itself. The ratio of the scales of the two figures will determine the time at which the encounter took place, while the plane of the cluster will fix the direction of relative velocity it encounters. This simple view of the matter, however, makes no allowance for the effects of internal motions of the stars in the cluster or for bending of the path of the cluster as a whole.

6. We can now examine what law of distribution of velocities is to be expected as the result of a single encounter between two clusters. Let the separate encounters of single stars be supposed to result in superposed velocities $u_1, u_2, \dots$ in the plane at right angles to $\bar{V}$. All directions in this plane are equally likely for the velocities $u_1, u_2, \dots$; let the angles they make with some fixed line in the plane be $\theta_1, \theta_2, \dots$. Then the resultant velocity V in the plane will be given by

$$V^2 = u_1^2 + u_2^2 + \dots + \Sigma 2u_1 u_2 \cos(\theta_1 - \theta_2),$$

so that the expectation of V^2 is Σu^2 .

The value of u is given by formula (2), so that the value of Σu^2 for all collisions which occur within a time t and for which σ lies between σ_1 (small) and σ_2 (large) is, from § 4,

$$\int_{\sigma_1}^{\sigma_2} \left(\frac{2m_2}{V\sigma} \right)^2 2v_2 V t \pi \sigma d\sigma = \frac{8\pi m_2^2}{V} v_2 t \log \left(\frac{\sigma_2}{\sigma_1} \right) \quad (4)$$

The value of m_1 , the mass of the star in the first cluster, has disappeared from this formula, so that it is true, no matter how much the masses of the stars in first cluster may vary *inter se*. If the masses in the second cluster vary, we may suppose $f(m)dm$ to express their law of distribution, and the true formula will be obtained from formula (4) on multiplying by $f(m)dm$ and integrating from $m=0$ to $m=\infty$. This simply requires us to interpret m_2^2 in formula (4) as indicating the mean value of m^2 for all the stars of the second cluster. Thus the product $v_2 m_2^2$ may be taken to be simply the value of Σm^2 taken in a unit volume of the second cluster. We may notice that a few heavy stars will contribute more to this quantity than a multitude of very small stars.

The integral (4) does not converge either for $\sigma_1=0$, or for $\sigma_2=\infty$. The reason for this is not difficult to find. Formula (3) only gave the true transverse velocity on the supposition that this velocity was itself small, and therefore that σ_1 was above a certain quantity; the formula had no applicability to very small values of σ . We must therefore, if possible, assign a finite limiting value to σ_1 , such that formula (3) is approximately true when $\sigma > \sigma_1$, and such also that when $\sigma < \sigma_1$ the star may be thought of as leaving the cluster altogether.

Similarly, a finite limit must be set for σ_2 ; our formulæ have no applicability to very great values of σ , for with the largest values of σ there would be a multitude of collisions happening simultaneously, and instead of the effects being additive, they would merely neutralise one another.

The best values to assume for σ_2 , σ_1 will depend on the particular problem in hand. For the universe as a whole, Eddington* takes the value $\log(\sigma_1/\sigma_2)=13.42$, and something comparable with this will probably be appropriate for most problems.

7. If c is the velocity of any one star in the plane of the disc, and if u , v are the components of this velocity along two axes Ox , Oy in this plane, it is clear that the law of distribution, being a function of c only, can be put in the form

$$\pi f(c^2) d(c^2)$$

and therefore in the form

$$f(c^2) du dv \dots \dots \dots (5)$$

It is also clear that there will be no correlation between u and v in the law of distribution, for these are both supposed to be so small in comparison with the relative velocity V as not to enter into the dynamics of collisions at all: stars having a small velocity v along Oy meet with just the same experiences with respect to their u -velocities as do stars having any other small velocity v_1 along Oy .

It follows that the law of distribution must be of the form

$$[\phi(u^2) du][\phi(v^2) dv],$$

and hence by comparison with (5), we must have

$$f(c^2) = \phi(u^2)\phi(v^2).$$

From this it follows, as in Maxwell's celebrated demonstration in the Theory of Gases,† that $f(c^2)$ must be of the form

$$f(c^2) = A e^{-kc^2} \dots \dots \dots (6)$$

where A and k are constants. Thus the velocities in the plane perpendicular to the relative velocity will obey Maxwell's law of distribution.

There are limits to the applicability of this law. It obviously cannot be true for very large values of c , since very violent collisions have been ruled out from our analysis. Neither can it be true for very small values of c , for in the case of stars in which forces of type (b) produce very little motion, the distribution of velocities must be entirely disturbed by the forces of

* *Stellar Movements*, p. 252.

† *Coll. Works*, ii. p. 380. The absence of correlation, which it is so difficult to prove in a gas, becomes a self-evident fact in the present problem.

type (*a*). But, to an approximation which will be true as regards order of magnitude only, we may suppose the expectation of $u^2 + v^2$ to be $1/k$. So that by comparison with formula (4), we obtain

$$\frac{1}{k} = \frac{8\pi m_2^2 v_2}{V} t \log \left(\frac{\sigma_2}{\sigma_1} \right) \quad (7)$$

8. In an actual problem there will doubtless be considerable variations from the law (6) produced by agencies which have been left out of sight in our simple investigation. Chief among these agencies will probably be the forces of type (*a*), those originating from other stars of the cluster. It is important to examine whether we can form some idea of the importance of these forces in comparison with those of type (*b*).

Let r be an average distance from star to star in the first cluster, defined by $v_1 r^3 = 1$. Then the force on any star in the first cluster arising from its nearest neighbour in the cluster will be of the order of magnitude of m_1^2/r^2 , where m_1^2 is of the order of magnitude of the average square of the mass of the stars in the first cluster. Let c_a^2 be the square of the velocity set up by this force of type *a*, in a time t , and let c_b^2 be the square of the corresponding velocity originating from the forces of type *b*. Then, as regards order of magnitude,

$$c_a^2 = \left(\frac{m_1}{r^2} t \right)^2 = \frac{v_1 m_1^2 t^2}{r}$$

$$c_b^2 = \frac{v_2 m_2^2}{V} t \left(8\pi \log \frac{\sigma_2}{\sigma_1} \right).$$

Let us fix our attention on an interval t such that $r = Vt$, this being roughly the interval in which a star in the second cluster passes from one star to the next in the first cluster. Then, for this interval, the ratio of c_a^2 to c_b^2 is

$$\frac{c_a^2}{c_b^2} = \frac{v_1 m_1^2}{v_2 m_2^2} \left(8\pi \log \frac{\sigma_2}{\sigma_1} \right) \quad (8)$$

Naturally enough by supposing $v_2 m_2^2$ to be very large compared with $v_1 m_1^2$, we can establish ideal conditions such that c_a^2 is negligible in comparison with c_b^2 . Thus the two special cases mentioned in § 2 are obtained by taking m_1/m_2 small and v_1/v_2 small respectively.

This covers the case of the scattering of a cloud of small stars by a cluster of big stars, but the more common case in nature will be one in which $v_1 m_1^2$ is approximately equal to $v_2 m_2^2$. When these quantities are exactly equal, the ratio (8) becomes $8\pi \log (\sigma_2/\sigma_1)$, and on assuming the value of σ_2/σ_1 given in § 6, this becomes equal to about 340.

Thus in our special interval t , the expectation of c^2 from forces of type *b* is about 340 times that from the largest of the forces of type *a*. But it must be remembered that there are

many forces of type a , and that the effects of these largely combine to reinforce one another. There will also be many intervals such as our special interval t , and in a sequence n of such intervals, n being so small that the forces of type b remain approximately the same, the effect of forces of type a is increased n -fold, while that of forces of type b is increased n^2 -fold. It is therefore clear that it will not in general be permissible to assume that the forces of type b outweigh those of type a .

9. Let us consider the special problem of a globular cluster. If Ω is the potential at any point, and c^2 the square of the velocity of a star relative to the centre of gravity of the cluster, the law of distribution of c is known to be of the form *

$$f(W) = f(\Omega - \frac{1}{2}c^2),$$

where $f(W)$ approximates to $W^{5/2}$ when W is small. The values of c^2 can only range from $c^2 = 0$ to $c^2 = 2\Omega$, and when Ω is small, it readily follows that the mean value of c^2 is $\frac{2}{3}\Omega$. When Ω is not small the mean value of c^2 remains comparable with Ω , but the exact ratio depends on the unknown law of distribution $f(W)$.

A first condition for the applicability of the foregoing analysis is that the values of c inside the cluster shall be small compared with the velocity of the cluster as a whole. The velocity of the cluster as a whole may be thought of as being from 20 to 60 Kms. a second. A velocity c of 1 Km. per second may, for our present purpose, be considered small compared with these proper motions, and will correspond to a value of Ω of about 10^{10} in C. G. units. Now six stars each of mass equal to that of our sun and at a distance away of 1 parsec. produce a potential of 7.2×10^7 , a second layer of 6×2^2 stars at 2 parsecs. distance will produce a potential of 14.4×10^7 , a third layer of 6×3^2 stars at 3 parsecs. distance will produce a potential of 21.6×10^7 and so on. The potential produced by 16 layers of this kind falls short of 10^{10} , although we have now built up a cluster of 17,953 stars at an average distance of less than 1 parsec. apart. Thus the condition in question is satisfied without difficulty except for very large or very close clusters. In an ordinary cluster the velocity of relative motion will be only a fraction of a kilometre per second.

The second condition, which concerns the ratio of c_a^2 to c_b^2 is less easily disposed of. At the edge of every cluster, however, v may be supposed to fall off as r^{-4} while Ω falls off as r^{-1} . Thus for sufficiently great values of r we have exactly the ideal conditions postulated in our analysis, and these conditions will be approximately fulfilled for some distance into the cluster. It is not easy to say for how far these conditions are satisfied, but a rough calculation shows that for a loose and small cluster they will be satisfied throughout the cluster. In general, we may

* See a paper "On the Law of Distribution in Star-Clusters" in this issue (page 567).

think of a cluster as consisting of an outer shell in which the conditions are satisfied, and an inner core in which they are not satisfied. The effect of an encounter on the outer shell will be to dissipate it in the way already explained. Some of its stars may move inwards and join the core, but those which acquire an outward motion will stream outwards in the plane normal to the direction of the relative velocities as though the core had been non-existent. The forces set up by the encounter and the loss of the outer shell both disturb the equilibrium of the inner core, and a new outer shell will be formed, which in its turn will also be dissipated. So the process goes on, the final result being a disc-shaped formation as in the simpler case.

10. I have tried to verify these conclusions by mathematical analysis. Suppose that a cluster is originally in equilibrium with a law of distribution

$$f(W)dudvdw=f[\Omega-\frac{1}{2}(u^2+v^2+w^2)]dudvdw,$$

and that the effect of an encounter is represented by the superposition of a Maxwellian distribution

$$Ae^{-k(u'^2+v'^2)}du'dv' \quad . \quad . \quad . \quad (9)$$

where k is very great so that the values of u' , v' , the components of the superposed velocity, are in general very small. The value of A must be chosen so that the integrated value of expression (9) is equal to unity; thus $A=k/\pi$.

The law of distribution resulting from this superposition will be

$$f(W)Ae^{-k(u'^2+v'^2)}du'dv'dudvdw \quad . \quad . \quad . \quad (10)$$

Let us transform from variables u, v, w, u', v' , to new variables U, V, w, u', v' , where U, V are given by

$$U=u+u', \quad V=v+v',$$

then the law of distribution (10) becomes

$$f[\Omega-\frac{1}{2}(U-u')^2-\frac{1}{2}(V-v')^2-\frac{1}{2}w^2]Ae^{-k(u'^2+v'^2)}dUdVdwdu'dv'.$$

The law of distribution of the components of resultant velocity, namely U, V, w , will be

$$F(U, V, w)dUdVdw,$$

where

$$F(U, V, w)=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}f[\Omega-\frac{1}{2}(U-u')^2-\frac{1}{2}(V-v')^2-\frac{1}{2}w^2]Ae^{-k(u'^2+v'^2)}du'dv'. \quad (11)$$

Since k is supposed large, the value of this integral will arise almost entirely from values of u', v' in the neighbourhood

of $u'=v'=0$. It will therefore be permissible to expand f , by Taylor's theorem, in the form

$$\begin{aligned} f[\Omega - \tfrac{1}{2}(U-u')^2 - \tfrac{1}{2}(V-v')^2 - \tfrac{1}{2}w^2] \\ = f(W) + u'Uf'(W) + v'Vf'(W) \\ + \tfrac{1}{2}u'^2[U^2f''(W) - f'(W)] + \tfrac{1}{2}v'^2[V^2f''(W) - f'(W)] + \dots \end{aligned}$$

where W now stands for $\Omega - \tfrac{1}{2}(U^2 + V^2 + w^2)$, and accents denote differentiation with respect to W .

Substituting this value for f in equation (11) and effecting the integrations, we obtain as the new law of distribution of velocities

$$F(U, V, w) = f(W) + \frac{1}{4k}[(U^2 + V^2)f''(W) - 2f'(W)] + \dots \quad (12)$$

the unwritten terms being of order k^{-2} and higher.

This law of distribution will not in general be one under which the cluster can remain in equilibrium, and it will proceed to rearrange itself under its own gravitational forces. In the absence of further encounters with external systems, the motion in this rearrangement will be determined by the equation *

$$\frac{\partial F}{\partial t} = -\frac{\partial \Omega}{\partial x} \frac{\partial F}{\partial U} - \frac{\partial \Omega}{\partial y} \frac{\partial F}{\partial V} - \frac{\partial \Omega}{\partial z} \frac{\partial F}{\partial w} - U \frac{\partial F}{\partial x} - V \frac{\partial F}{\partial y} - w \frac{\partial F}{\partial z}.$$

The initial stages of the motion will be obtained by substituting on the right the value of F given by equation (12). The equation is then found to assume the simple form

$$\frac{\partial F}{\partial t} = -\frac{1}{2k} \left(U \frac{\partial \Omega}{\partial x} + V \frac{\partial \Omega}{\partial y} \right) f''(W) \quad (13)$$

At first the law of distribution will be of the form $F_0 + F_1 t$, where F_0 is given by equation (12) and F_1 by equation (13), but this law will fail as soon as t^2/k becomes appreciable.

Multiplying both sides of equation (13) by $dU dV$, and integrating over all values of U, V , we obtain

$$\frac{\partial}{\partial t} \iint F dU dV = 0.$$

But $\iint F dU dV$ expresses the law of distribution of the w -component of velocity. Thus we obtain the result that, within the time just mentioned, the law of distribution of w -components is unchanged. Neither from the encounter, nor from the earlier stages of the subsequent rearrangement of the cluster is there any tendency for velocities to be set up normal to the plane of the cluster.

We notice that the right-hand member of equation (13) is

* *M. N.*, lxxiv. p. 71.

invariant, as it ought to be, with regard to a change of axes in the plane of xy , so that the motion in the plane xy is the same along all radii. It is accordingly sufficient to consider only the motion on the positive axes of x . Here $\partial\Omega/\partial x$ is negative. The general value of $f(W)$ is known to be

$$f(W) = W^{5/2} [a_0 + a_1 W + a_2 W^2 + \dots] \quad (14)$$

so that the most general value for $f''(W)$ is

$$f''(W) = \frac{1}{4} W^{\frac{1}{2}} [3.5a_0 + 5.7a_1 W + 7.9a_2 W^2 + \dots] \quad (15)$$

At the outer edge of the cluster $f(W)$ reduces to $a_0 W^{5/2}$, so that a_0 must be positive. In the same region $f''(W)$ reduces to $\frac{1.5}{4} W^{\frac{1}{2}} / 2a_0$, so that this again is positive. It is not, however, possible to prove that $f''(W)$ is positive throughout the cluster; we must be content with the statement that it is positive from the edge to within a certain unknown distance from the centre.

Throughout the shell in which $f''(W)$ is positive, equation (13) assumes the form on the positive axis of x ,

$$\frac{\partial F}{\partial t} = U \times \text{a positive quantity,}$$

so that F increases for positive values of U , and decreases for negative values. Thus in the process of readjustment after the encounter, continually more stars tend to stream outward in the plane of xy and continually fewer tend to stream inward. The result has, strictly speaking, been proved only for very large values of k and for the initial stages of the motion of readjustment. It may therefore be regarded as a confirmation of the general tendency discovered in § 9, but it has not provided a general proof.

11. Let us now proceed on the supposition that, to an approximation which will be good or bad according to circumstances, the law of distribution of the u, v velocity components after one encounter is that given by equation (6), namely,

$$Ae^{-k(u^2+v^2)} du dv \quad (16)$$

If the dimensions of the cluster before the spreading out were small in comparison with its dimensions after a time T , then the law of star-density after time T will be approximately

$$AT^{-2} e^{-k(x^2+y^2)T^{-2}} dx dy \quad (17)$$

In an actual problem the distribution will not be exactly in a plane, and the cluster will be of the shape of a lens or biscuit rather than of a disc. But, to an approximation which may be good or poor, the law of distribution of stellar distances from the centre will be

$$Bre^{-hr^2} dr \quad (18)$$

where $B = 2\pi A/T^2$ and $h = \kappa/T^2$. This law is one which is

familiar through having been suggested as giving the law of stellar distances in our own universe.*

12. The essential result of a single encounter has been seen to be a spreading out into a disc, which will continue to expand as time goes on. The effect of a second encounter superposed on to the first can be calculated, and the requisite analysis is not difficult in simple cases. To take the easiest instance of all, imagine that a cluster A first meets a cluster B in an encounter with relative velocity parallel to Oz , and that after, or even simultaneously with, this encounter, it meets a similar cluster C with a relative velocity parallel to Ox . The resulting law of distribution of velocities will be obtained by superposing two similar Maxwellian distributions on to the same set of stars.

Let the two distributions separately be supposed to be

$$Ae^{-\kappa(u_1^2+v^2)} du_1 dv \text{ and } Ae^{-\kappa(u_2^2+w^2)} du_2 dw,$$

where u_1, u_2 are the contributions to the u -component of velocity from the two distributions separately. Write

$$u_1 + u_2 = u, \quad u_1 - u_2 = u'.$$

Then the superposition of the two distributions gives the distribution

$$\begin{aligned} & A^2 e^{-\kappa(u_1^2 + u_2^2 + v^2 + w^2)} du_1 du_2 dv dw \\ &= \frac{1}{2} A^2 e^{-\frac{1}{2}\kappa(u^2 + u'^2) - \kappa(v^2 + w^2)} du du' dv dw. \end{aligned}$$

Integrating with respect to u' , we obtain for the law of distribution of u, v, w ,

$$Be^{-\kappa(\frac{1}{2}u^2 + v^2 + w^2)} du dv dw \quad . \quad . \quad (19)$$

Thus, in this simple case, the law of distribution in the cluster after the double encounter has taken place is of Schwarzschild's ellipsoidal form, the ratio of the axes of the ellipsoid being $\sqrt{2} : 1 : 1$.

The directions of the axes ought to be noticed. After cluster A has passed through cluster B with relative motion in the direction Oz , and cluster C with relative motion in the direction Oy , an astronomer in cluster A will observe star-streaming in the direction Ox .

13. More generally (but always with the limitations of the simple assumptions on which we are working), the law of distribution in a cluster after any number of encounters with other clusters will be obtained by compounding a number of two-

* See Dyson (*M.N.*, lxxiii. p. 334), Eddington (*M.N.*, lxxiii. p. 530), Unthank (*M.N.*, lxxvi. p. 531). Dyson finds the law gives good results for stars of magnitude brighter than 10.3; Eddington for stars brighter than 5.8. It has not, however, as Professor Eddington has pointed out to me, been either suggested or tested as a law for stars of all magnitudes.

dimensional Maxwellian distributions. It is easy to prove that the resulting law must necessarily be of the ellipsoidal form, although there is naturally no longer any simple relation between the axes.

The result of superposing two ellipsoidal distributions, having for simplicity their axes parallel, say

$$e^{-\left(\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2}\right)} \text{ and } e^{-\left(\frac{u'^2}{a'^2} + \frac{v'^2}{b'^2} + \frac{w'^2}{c'^2}\right)}$$

is to obtain a new ellipsoidal distribution

$$e^{-\left(\frac{u^2}{a^2+u'^2} + \frac{v^2}{b^2+v'^2} + \frac{w^2}{c^2+w'^2}\right)},$$

so that in this simple case the law for compounding is to add squares of corresponding semi-axes. Hence on compounding a number of two-dimensional Maxwellian distributions, all supposed for simplicity to be in the principal planes xy , yz , or xz , we obtain an ellipsoidal distribution in which the squares of the semi-axes are of the form

$$a^2 = r + s, \quad b^2 = s + q, \quad c^2 = q + r,$$

where q , r , s are positive quantities. The largest semi-axis a can never be greater than $\sqrt{(b^2 + c^2)}$, but except for this limitation, a , b , and c can have any values, and this can be seen to remain true when the constituent distributions are not supposed limited to the three orthogonal planes.

It is therefore doubtful whether it is possible in this simple way to obtain an ellipsoidal distribution with axes equal to those in the observed distribution of our universe.*

14. If a cluster meets a great number of other clusters, and the relative velocities at encounter are arranged at random, the final law of distribution will be obtained by compounding two dimensional Maxwellian distributions with their planes arranged at random. The result is obviously a spherical Maxwellian distribution.

Under natural conditions these planes will not be arranged at random, but will show a preference for directions at right angles to the direction of motion of the cluster. Obviously this will lead to a spheroidal law of distribution, the spheroid being oblate, and having the direction of motion of the cluster for its axis of revolution.

So also a cluster passing through a chaos of independent and random-moving stars will acquire a spheroidal law of distribution, and in time an approximately spheroidal shape. The ratio of the axes will depend on the ratio of the velocity of the cluster to the average velocity of the stars. For a very rapidly moving cluster the spheroid will be very flat; for a slowly moving cluster it will be nearly spherical.

* Eddington (*Stellar Movements*, p. 125) gives five estimates for the ratio of the axes, assuming $b=c$, and these range from 2.13:1:1 to 1.59:1:1. Gyllenberg (*Lund Meddelanden*, ii. 13) freeing himself from the assumption that $b=c$, finds $a:b:c=4.20:3.47:2.67$, so that $a^2 < b^2 + c^2$.

15. Laws of density cannot be compounded in the same simple way as laws of distribution of velocity. For the final law of density in a cluster will depend not only on the encounters experienced, but also on the times at which they occurred. To illustrate this, let us return to the simple problem of § 12, in which a cluster A meets in turn two other clusters B and C.

Suppose first that the encounters with the two clusters B and C occur almost simultaneously, and that before this the cluster A has been quite compact. After the encounters are over, the cluster A will become simply a material representation of its own velocity-diagram, and when it has expanded to a size large compared with its original dimensions, it will be a prolate spheroid with axes in the ratio $\sqrt{2}:1:1$. In this spheroid the proper motions at every point will be radial.

On the other hand, suppose that the encounter with cluster B occurs first, and that the encounter with cluster C only happens after a long interval. During this interval the cluster will have assumed a disc-like form in the plane of xy , and will be expanding in this plane, its law of star-density being approximately that given by formula (17). After the second encounter has taken place, the law of distribution of velocities will at once be changed to the ellipsoidal law found in § 12 (formula (19)), but the law of star-densities will at first be still that given by formula (17). Gradually, of course, this law will change, and the cluster will assume a more ellipsoidal form. But shortly after the second encounter, the system will have the shape of a somewhat thickened disc or lens of approximately circular shape, the law of stellar densities being approximately that given by formula (18), namely

$$B e^{-hr^2} dr,$$

while the law of distribution of velocities will be of the ellipsoidal form found in § 12,

$$e^{-k(\frac{1}{2}u^2+v^2+w^2)} du dv dw,$$

and star-streaming will appear to take place along the axis of x , a direction lying in the plane of symmetry of the cluster.

In this way it is possible to construct a primitive model of our own system, but such a model will only be acceptable to those who are willing to admit the possibility of our galactic system being one out of many similar systems. We must, in fact, think of the whole system as only one "molecule" in a gas of many molecules.*

16. For simplicity we have always assumed that the cluster

* A condition for our analysis to apply will be that the velocity of our sub-universe in space shall be large compared with the relative motions of stars inside this sub-universe. For the velocity of our sub-universe in space, Young and Harper (*Journ. Royal Ast. Soc. of Canada*, 10, p. 134) find 598 kms. per sec., while H. O. Truman (*The Observatory*, 39, p. 134) suggests the nearly equal value of 670 kms. per sec. For purposes of our analysis, velocities of this order are certainly large in comparison with relative velocities inside our sub-universe.

whose fortunes we have been following has been the smaller in every encounter, and that it has been completely enveloped by the larger cluster. We must now look at the effect of an encounter on the larger cluster.

Consider a flat cluster of area A , and suppose that an area a of this is affected by the passage through it of a smaller cluster of cross-section a . The effect of the passage of this smaller cluster on the stars originally occupying the area a will be to generate in them velocities transverse to the direction of relative velocity and so to dissipate these stars. Thus the area a will first appear as a region in which the star-density is below that in the surrounding area $A-a$, and afterwards as a hole in the cluster A . Doubtless there will be other tendencies at work, some of them acting in the direction of neutralising this effect, but for the present we must be content to disregard them, and confine our attention to the one tendency that we are trying to investigate. And the final effect of this one tendency alone would be to produce a gap in the cluster. Viewed edgewise, the cluster would seem to be emitting streams of stars from the area a , while viewed normally the area a would look like a dark patch on the bright area A . Viewed by an astronomer on a star remaining inside the area a , the surrounding area $A-a$ would look like a dense galaxy of stars encircling a lens-shaped universe of more scattered stars.

17. In conclusion, it must again be repeated that the problem we have had under discussion has been far too complex for anything like a full solution to be obtained. In our brief investigation it has repeatedly been forced upon us that there must be many agencies at work simultaneously in the processes we have been considering, while we have confined our attention to one. It is as though a rich pattern were being woven, and we had only been able to follow one thread. The question at issue is whether, by following this one thread, we have been able to form any conception, even the most incomplete, of the weaving of the whole pattern. It seems to me that the figures traced, so to speak, by the one thread we have followed have shown some resemblance to those which appear in the complete picture disclosed by observational astronomy. For this reason I am inclined to suggest, although only tentatively and conjecturally, that this one thread may really provide a clue to some of the processes which are at work in the weaving of the whole.

On the Law of Distribution in Star-Clusters.

By J. H. Jeans, M.A., F.R.S.

1. In a recent paper* it was shown that a cluster of stars, free from all external influences, can be in a steady state in only two cases. In the first case, star-streaming occurs, and the

* *M.N.*, lxxvi. p. 70.

figure is not necessarily spherical: Saturn's rings provide an example of a steady state of this type. In the second case, there is no star-streaming, and the cluster is necessarily spherical in shape. Illustrations of this type of steady motion may be looked for in globular star-clusters. We proceed to consider the conditions of equilibrium for such clusters.

2. Let Ω be the gravitational potential at x, y, z , and let u, v, w be the components of velocity of a star at x, y, z . Put $u^2 + v^2 + w^2 = c^2$, and put

$$W = \Omega - \frac{1}{2}c^2,$$

so that W is the negative energy of the star. As the star describes its orbit W will remain constant, and the condition that the star shall not pass out of the attraction of the cluster is that W shall be positive. For a permanent cluster W must be positive for every star.

The law of distribution must be of the form *

$$f(W)du dv dw dx dy dz,$$

so that the star-density at any point will be

$$\nu = \iiint f(W) du dv dw = 4\pi \int f(W) c^2 dc, \quad (1)$$

the limits of integration for c being from $c=0$ to $W=0$.

Put $c^2 = 2\Omega \sin^2 \theta$, then the limits for c are from $\theta=0$ to $\theta=\frac{1}{2}\pi$, and equation (1) becomes

$$\nu = 4\pi (2\Omega)^{\frac{3}{2}} \int_0^{\frac{1}{2}\pi} f(\Omega \cos^2 \theta) \sin^2 \theta \cos \theta d\theta \quad (2)$$

The solutions of this equation, combined with $\nabla^2 \Omega = -4\pi m\nu$, will give the different possible configurations for star clusters in which no streaming takes place.

3. At infinity the value of Ω will be of the form M/r , where M is the total mass of the cluster. Let us write u for $1/r$, and assume an expansion for Ω in the form

$$\Omega = Mu(1 + c_1 u + c_2 u^2 + c_3 u^3 + \dots) \quad (3)$$

In general, the equation $\nabla^2 \Omega = -4\pi m\nu$ assumes the form

$$u^4 \frac{d^2 \Omega}{du^2} = -(4\pi)^2 m (2\Omega)^{3/2} \int_0^{\frac{1}{2}\pi} f(\Omega \cos^2 \theta) \sin^2 \theta \cos \theta d\theta \quad (4)$$

At infinity ($u=0$), the limiting form of this equation is

$$2Mc_1 u^4 = -(4\pi)^2 m (2Mu)^{3/2} \int_0^{\frac{1}{2}\pi} f(Mu \cos^2 \theta) \sin^2 \theta \cos \theta d\theta.$$

It follows that when $u=0$, the limiting form of

$$\int_0^{\frac{1}{2}\pi} f(Mu \cos^2 \theta) \sin^2 \theta \cos \theta d\theta$$

* L.c. p. 79.

must be proportional to $u^{5/2}$, and hence that the limiting form of $f(x)$ when $x=0$, must be proportional to $x^{5/2}$.

It is now readily seen that the correct form to assume for the general solution of equation (4) is

$$f(\Omega \cos^2 \theta) = (\Omega \cos^2 \theta)^{5/2} [a_0 + a_1 \Omega \cos^2 \theta + a_2 (\Omega \cos^2 \theta)^2 + \dots] \quad (5)$$

With this solution, equation (2) gives

$$\begin{aligned} \nu &= 4\pi (2\Omega)^{\frac{3}{2}} \int_0^{\frac{3}{2}\pi} \sum_0^{\infty} a_p (\Omega \cos^2 \theta)^{p+\frac{5}{2}} \sin^2 \theta \cos \theta d\theta, \\ &= (2\pi)^{\frac{3}{2}} \sum_0^{\infty} a_p \Omega^{p+4} \Gamma(p+\frac{7}{2}) / \Gamma(p+5) \end{aligned} \quad (6)$$

so that in place of equation (4) we have

$$Mu^4(1.2c_1 + 3.2c_2u + 4.3c_3u^2 + \dots)$$

$$= -(2\pi)^{\frac{3}{2}} \sum_0^{\infty} a_p (Mu)^{p+4} (1 + c_1u + c_2u^2 + c_3u^3 + \dots)^{p+4} \Gamma(p+\frac{7}{2}) / \Gamma(p+5).$$

Equating the coefficients of the various powers of u , we obtain

$$c_1 = -5\pi^2 M^{\frac{3}{2}} 2^{-\frac{1}{2}} a_0, \quad (7)$$

$$3c_2 - 4c_1^2 = -7\pi^2 M^4 2^{-\frac{1}{2}} a_1, \text{ etc.} \quad (8)$$

Clearly we can obtain unique values of $a_0, a_1, \dots$ corresponding to any values we choose to assign to $c_1, c_2, \dots$. To get a physical solution of our problem, $c_1, c_2, \dots$ must be such that ν is positive and finite at every point of the cluster, and there is the further restriction that the resulting values of $a_0, a_1, \dots$ must be such that $f(W)$ is positive for all possible values of W . But, without violating these restrictions, it is clear that there will be a multiply infinite set of values for $c_1, c_2, \dots$ and corresponding sets of values for $a_0, a_1, \dots$. If it were possible to observe the law of distribution of density accurately for a cluster, we could determine the law of distribution of velocities from equations (7), (8), etc., and it might be possible to make some deductions as to the past history of the cluster.

4. The equations assume their simplest form when only one of the coefficients $a_0, a_1, a_2, \dots$ exists, say a_p . In this case the equation satisfied by Ω reduces to

$$\nabla^2 \Omega = u^4 \frac{\partial^2 \Omega}{\partial u^2} = -2(2\pi)^{\frac{3}{2}} m \frac{\Gamma(p+\frac{7}{2})}{\Gamma(p+5)} a_p \Omega^{p+4}.$$

Since a_p must from its meaning be positive, this may be put in the form

$$\nabla^2 \Omega + \alpha^2 \Omega^{p+4} = 0 \quad (9)$$

This is the differential equation satisfied by a certain power of the density in a gaseous spherical nebula in which the gas is

supposed to obey the adiabatic law. As such it has been very fully studied by Emden.*

The simplest solution is found to occur when

$$a_0 = 0; a_2 = a_3 = a_4 = \dots = \infty,$$

so that $\nabla^2 \Omega$ is proportional to Ω^5 . The simplest solution for ν is found to be

$$\nu = \frac{A}{\left(1 + \frac{a^2}{r^2}\right)^{\frac{5}{2}}} \quad \dots \quad (10)$$

in which A and a are constants of integration. This particular solution was first given by Schuster in 1883.† A more general integral of the same equation, in which ν is not limited to being finite and continuous at the centre, has been found by v. Zeipel.‡ These two laws have been shown by Plummer§ and v. Zeipel|| to represent with a good deal of accuracy the observed distribution in a number of clusters. It is found that the law may fail in two regions—the centre of the cluster and its outer edges. The failure in the centre, when it occurs, may perhaps be attributed to the existence of $a_2, a_3, a_4, \dots$; the failure at the outer edge, when this occurs, to the existence of a_0 .

5. There is no apparent reason why a_0 should tend to vanish, so that we should expect that in general a_0 will have a finite value, and hence, from equation (7), that c_1 will also have a finite value. The law of distribution in the outermost regions in the cluster ought therefore in general to be of the form

$$\nu = k/r^4$$

where the value of k is found to be $a_0 M / 2\pi m$, or $a_0 N / 2\pi$, where N is the number of stars in the cluster, supposed all equal. It follows that the law of distribution on a photographic plate ought, as regards the outermost regions, to be of the form cr^{-3} . In the following tables, n stands for the density of stars observed at a distance r from the centre on plates of the clusters in question, the material being taken from Plummer's¶ discussion of Bailey's counts** of these clusters. The product nr^3 is found to be approximately constant for each cluster for fairly large values of r . I have taken an average value of this product to give the value of c , and have calculated the value of cr^{-3} for comparison with n . For eight out of Plummer's ten clusters, the observed values of n seem to agree very much better with the formula cr^{-3} , than with Plummer's formula derived from

* R. Emden, *Gaskugeln: Anwendungen der Mechanischen Wärme-theorie*: especially chaps. IV-VI.

† *Brit. Assn. Report*, 1883, p. 428.

‡ *K. Svenska Vetensk. Acad., Handl.*, Bd. li. No. 5.

§ *M.N.*, lxxvi. p. 107.

|| *L.c.*

¶ *M.N.*, lxxvi. p. 107.

** *Harvard Coll. Obs. Annals*, 76, No. 4.

equation (10). For the two clusters III. and IV, for which no tables are given here, the agreement is less good.

I. N.G.C. 362 ($c=1110$).			II. N.G.C. 5272 ($c=2450$).		
r	cr^{-3}	n	r	cr^{-3}	n
11	0.83	0.9	11	1.8	1.7
12	0.64	0.7	12	1.4	1.4
13	0.50	0.5	13	1.1	1.2
14	0.40	0.4	14	0.90	1.0
15	0.33	0.3	15	0.73	0.8
16	0.27	0.2	16	0.60	0.6
17	0.22	0.2	17	0.50	0.5
18	0.19	0.2	18	0.42	0.4
19	0.16	0.1	19	0.35	0.3
20	0.14	0.1	20	0.31	0.2

V. N.G.C. 6266 ($c=2300$).			VI. N.G.C. 6809 ($c=2000$).		
r	cr^{-3}	n	r	cr^{-3}	n
6	10.7	9.6	15	0.59	0.6
7	6.7	7.1	16	0.49	0.5
8	4.5	5.0	17	0.41	0.4
9	3.2	3.4	18	0.34	0.3
10	2.3	2.3	19	0.29	0.3
11	1.7	1.5	20	0.25	0.2

VII. N.G.C. 7078 ($c=1080$).			VIII. N.G.C. 7089 ($c=1260$).		
r	cr^{-3}	n	r	cr^{-3}	n
6	5.0	5.5	6	5.8	6.0
7	3.2	3.0	7	3.7	3.7
8	2.1	2.0	8	2.4	2.4
9	1.5	1.5	9	1.7	1.7
10	1.1	1.1	10	1.3	1.1
11	0.81	0.8	11	0.93	0.8
12	0.62	0.6	12	0.72	0.5
13	0.49	0.5	13	0.57	0.3
14	0.39	0.4	14	0.45	0.2
15	0.32	0.3	15	0.37	0.1
16	0.26	0.2	16	0.30	0.0
17	0.22	0.2			
18	0.19	0.2			
19	0.16	0.1			
20	0.13	0.1			

IX. N.G.C. 7099 ($c=520$).			X. N.G.C. 104 ($c=2650$).		
r	cr^{-3}	n	r	cr^{-3}	n
6	2.4	2.5	17	5.4	5.8
7	1.5	1.6	18	4.5	4.9
8	1.0	1.0	19	3.9	4.1
9	0.71	0.7	vi	3.3	3.3
10	0.52	0.6	vii	2.1	2.1
11	0.39	0.4	viii	1.4	1.4
12	0.30	0.3	ix	0.98	0.9
13	0.24	0.2			
14	0.19	0.2			
15	0.15	0.1			
16	0.13	0.1			

For purposes of detailed comparison I have taken my table of cluster VII as a sample. Plummer states that the agreement of n in the outer regions with his formula is "excellent" for this particular cluster. The errors from $r=16$ to the end would be large with any formula, since obviously the given values $n=0.2$ and $n=0.1$ are themselves very inaccurate. Omitting these values, I find for the range $r=6$ to $r=15$, that the error in my table is less than that of Plummer's table in seven cases, and is equal in the remaining three. The average of the errors for these ten entries is 11.9 per cent. in Plummer's table, and is 3.3 per cent. in mine. Other comparisons give somewhat similar results.

There seems, then, to be little doubt that the density in the normal globular cluster falls off as $1/r^4$ at great distances from the centre. Beyond this it does not seem possible to pick out any definite features in the law of distribution which are common to all clusters. This, however, is in accordance with the theory given in §§ 2 and 3 of this paper.

The Distribution of Stars in Globular Clusters. By A. S. Eddington, M.A., M.Sc., F.R.S., Plumian Professor.

1. The object of this paper is to discuss the significance of the law of density $\rho = (c^2 + r^2)^{-\frac{5}{2}}$, which has been found by H. C. Plummer* and H. von Zeipel† to govern the distribution of stars in globular clusters.

Investigators have usually made a comparison between the distribution of stars in globular clusters and the distribution of

* *M.N.*, vol. lxxi. p. 460, and vol. lxxvi. p. 107.

† *K. Svenska Vetensk. Acad., Handl.*, Bd. li. No. 5.

mass in a sphere of gas in equilibrium under its own gravitational attraction. A variety of forms for comparison is obtained, since the gas takes different forms of adiabatic equilibrium according to the value of γ , the ratio of the specific heats; isothermal equilibrium can also be included by setting $\gamma=1$. Plummer's law was derived by taking $\gamma=1.2$; and it was shown by him that this distribution agrees very satisfactorily with the observed distribution, except that in certain cases the law appears to break down at the centre of the cluster.

It is not evident from Plummer's investigations how far γ could deviate from the value 1.2 without the agreement being sensibly spoiled. Individual discussions by Plummer, See and von Zeipel, however, showed that the values $\gamma=2$, 1.57 and 1, respectively, failed to give proper agreement. More recently v. Zeipel has found that the margin of alteration of γ must be very small indeed. It is rather more convenient to set

$$\gamma = 1 + 1/n,$$

so that $n=5$ gives Plummer's formula. By a least-squares solution v. Zeipel found that the values of n which best fit the four clusters *M.* 2, 3, 13, 15 are respectively $n=5.15$, 5.05, 4.92, and 5.08. The corresponding values of γ range from 1.194 to 1.203. The agreement of these values is rather too good, since the probable errors of n range from ± 0.26 to ± 0.61 .

It was found by Plummer that in many clusters the central region (within about 30" of the centre) is more dense than the theoretical formula allows. The excess is sometimes two- or threefold, and may represent 100 or more stars. Whilst noting the possibility that the disagreement may be due to the difficulty of counting the densest part, Plummer seems to regard the excess as probably real, so that the adiabatic law breaks down there. It is well to remember that in that case some care must be exercised in interpreting the agreement of the outer parts of the cluster with the law $\rho = (c^2 + r^2)^{-\frac{5}{2}}$, because this law has no theoretical basis when applied to part only of a cluster. We cannot add stars at the centre without altering the equilibrium of the whole system. In the cluster *M.* 15 (the extreme case) there is a central excess of 270 stars superposed on the theoretical distribution of 926 stars. The attraction of the former must alter very considerably the equilibrium distribution of the outer stars. Von Zeipel in his investigation allowed for the doubtful conditions at the centre by using a more general formula which includes a parameter a , depending on the unknown mass near the centre. The values of a were found in his least-squares solution, and in all four clusters they are less than the normal value (*i.e.* when the central mass conforms with the rest). I find that this corresponds to a central *excess* of density. It is a satisfactory vindication of the theory that v. Zeipel from his examination of the outer parts could

predict a central excess of density which Plummer has observed directly.

2. The methods and tables of v. Zeipel are especially convenient for testing the adiabatic or other proposed laws. The observed counts relate to the apparent density σ projected in two dimensions, and v. Zeipel has shown how to obtain from these the true three-dimensional density ρ at a distance r from the centre. Incidentally it may be remarked that ρ depends on σ only through $d\sigma/dr$, so that ρ is independent of any constant added to σ ; thus there is no need to estimate an allowance for the non-cluster stars present in the region and forming the general stellar background. In general, the theoretical laws with which we wish to compare the observed distribution are given as relations between the density ρ and potential ϕ ; the relation between ρ and r is only expressible as an intractable differential equation. The most convenient procedure is therefore to deduce ϕ from ρ in the *observed* distribution, which can be done without difficulty by a double quadrature, since

$$-4\pi\rho = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right).$$

It may be necessary to add to ϕ the complementary function $A+B/r$ where A and B are uncertain within limits; but it is easy to arrange the tests so as to eliminate A and B . Having then obtained a series of corresponding values of r , ρ and ϕ for the observed distribution, we can easily test any suggested theoretical law. The chief of these are—the adiabatic laws $\rho=\phi^n$, the isothermal law $\rho=e^\phi$, and the law deduced by the writer for a globular stellar system $\rho=e^\phi/(1+r^2)$.

A little experience in attempting to fit other laws to v. Zeipel's numbers has convinced me that the adiabatic law with $n=5$ is immensely superior to any other. It may be only a first approximation; but it seems probable that $n=5$ has some fundamental significance in connection with the structure of star clusters.

3. *Criticism of other Theories.*—Plummer suggests that the distribution of stars in a globular cluster merely reflects the distribution of matter in the nebula from which it was formed, and that $\gamma=1.2$ may well represent the average ratio of the specific heats of the primitive gases. But, as v. Zeipel has remarked, it seems scarcely possible that this distribution could be conserved during the formation of the stars; the stars cannot crystallise out instantaneously; and when no longer supported by the gaseous pressure the primitive distribution of mass would rapidly change.

Von Zeipel, on the other hand, explains the law by supposing that the stars themselves behave like the molecules of a gas, and are therefore distributed according to the laws applying to gases. Although it seems clear that the gas-theory is inapplicable to the stellar universe, it may have more importance in star-clusters. But against von Zeipel's theory I have three objections.

(1) *The law ought to be that of isothermal equilibrium—not adiabatic.*—Adiabatic equilibrium in a gas is due to convection currents, caused either by heating from an external source or radiation of energy from the outer layers into space. In a star cluster there is no radiation of the energy of stellar motions, and there seems to be no possibility of convection currents arising. It is true that stars will continually pass from the inside to the outer layers and *vice versa*; but this does not correspond to a convection current. It is essential to the adiabatic law that, when the potential energy of a mass of gas is increased, the energy comes not from the mass-motion—*i.e.* the convection current—but from the heat-energy or internal motions of the mass. But if a number of stars are carried by their own motions to a place of greater potential energy, this energy is gained at the expense of their common mass-motion, not of their internal or relative motions.

(2) *If the equilibrium is adiabatic, we should have $\gamma = 1.67$, the value for monatomic molecules.*—Von Zeipel meets this by supposing that a large proportion of the stars are binary systems which behave like more complex molecules. But in order that a binary system should behave like a complex molecule the energy of orbital motion must be available to be drawn on to supply potential energy to the star. When a “diatomic” star passes from the centre to the outside of the cluster, two-fifths of the gain of potential energy must be drawn from the orbital motion. The orbit will therefore expand and contract as the binary approaches and recedes from the centre. This appears to me fantastic and incredible.

(3) *Even for double stars the value of γ would not be brought below 1.40.*—Von Zeipel answers this by taking into account other degrees of freedom in binary systems, corresponding to the ellipticity of the orbit. In order to obtain the value 1.20 he finds it necessary to attribute great density and great velocities (of the order 100 kms. per sec.) to the cluster. Leaving aside the question of whether such a constitution is probable, it would appear on this hypothesis that the value 1.20 is an accident arising from the particular density and proportion of double stars in the cluster. There seems no reason why it should occur so uniformly in clusters differing widely in richness and character.

4. The constant $\gamma = 1.20$ regarded as the ratio of specific heats seems singularly arbitrary. It corresponds to 7 internal degrees of a freedom, a number which can scarcely be in any way significant. From this aspect its only importance is that it leads to the one case in which the density in a gas-sphere can be integrated in finite terms—a property which is useful, but obviously irrelevant.

But the case is different when, instead of interpreting γ physically, we regard γ as a mathematical constant in the equation for the density. The value $\gamma = 1.2$, or $n = 5$, is an

important critical value.* For $n < 5$ the gas is in equilibrium in a globular form having a finite boundary. For $n > 5$, the mass of the distribution is infinite; in other words a finite mass of gas cannot exist in this form if $n > 5$. For the critical case $n = 5$ the matter extends to infinity, but the mass is finite. Our clusters therefore take up the most diffused form of equilibrium possible for a finite number of stars.†

It seems much more likely that the value of n is determined by this property rather than by any possible interpretation in connection with specific heats.

5. I propose now to apply to this problem methods similar to those used by J. H. Jeans and myself in discussing stellar systems in a steady state. The results of those investigations are applicable if the stars move practically undisturbed by chance encounters for several revolutions at least; on any probable hypothesis as to the density in clusters this would be satisfied.

If the motions at any point are according to the Maxwellian law—a state which might be brought about by gradual modification of the orbits by encounters during a very long past history—the only possible steady state is the isothermal law‡ $\rho = e^{2\phi}$. This law is widely different from that actually found, and cannot represent the density.

Another possibility is that there may be star-streaming in a radial direction; we can then have the law $\rho = e^{2\phi}/(1+r^2)$ which I have discussed at length in a former paper. I have tested this law by comparison with von Zeipel's table for Messier 3. The divisor $(1+r^2)$ helps to make the agreement better than in the case of the isothermal law; but the agreement is poor compared with Plummer's law.

The above, and any other laws which provide no upper limit to the velocities, involve an infinite mass. For a finite system any star whose velocity exceeds $\sqrt{2\phi}$ escapes from the system, and therefore cannot form part of a steady distribution. This suggests that we should truncate the foregoing distributions by cutting out all stars which on the Maxwellian or ellipsoidal law have velocities above this limit. Since any star is permanently either above or below the limit, the exclusion does not upset the steady state; though, of course, the alteration of density involves an entire readjustment of the system.

The modified Maxwellian law then gives $\rho = e^{2\phi} \frac{4}{\sqrt{\pi}} \int_0^{\sqrt{2\phi}} e^{-v^2} v^2 dv$. This represents a density which near the centre varies as $e^{2\phi}$ and at the outside as $\phi^{\frac{3}{2}} e^{2\phi}$. It is a slight improvement on the

* The properties of these systems are fully discussed in Emden's book *Gaskugeln*.

† Provided the system is limited to one of the forms $\rho = \phi^n$.

‡ In the Second Paper, *M.N.*, vol. lxxv. p. 370, equation (14), we must put $k=0$, since by hypothesis there is no star-streaming. By suitable choice of units the expression then reduces to $\rho = e^{2\phi}$.

original law, but does not satisfy the observations. The modification of the ellipsoidal distribution is more complicated, but I believe that it also is unsatisfactory.

These "truncated" distributions would be very unstable, because they contain a considerable number of stars on the verge of escape. If a few stars escaped by accident they would probably be followed by others, since ϕ would be lowered through part of the cluster, and stars originally safe would be left with velocities above the limit. It seems that a more thoroughgoing modification of the Maxwellian law is required. The error law may be expected whenever the velocities are determined by a large number of small independent causes acting in random directions, and we should therefore expect the smaller velocities to follow the error-law approximately; but the absence of velocities greater than $\sqrt{(2\phi)}$ will disturb the conditions, and the larger velocities will be, as it were, drained away as we approach this limit. The law of distribution should therefore be approximately the Maxwellian law for small velocities, but modified for the larger velocities so as to fall to zero at or before the limit $\sqrt{(2\phi)}$.

6. The treatment of systems in which the velocity-law deviates from the law of errors is greatly simplified by a theorem given by Jeans.*

If the number of stars N in an element $dx dy dz$ with velocities within a range $dx' dy' dz'$ be $f dx dy dz dx' dy' dz'$, then f must be a function of the first integrals of the equations of motion. For globular symmetry of distribution, if v be the transverse velocity, u the radial velocity, w the total velocity, and $V^2 = 2\phi$, this gives

$$N = 2\pi f(V^2 - w^2, r^2 v^2) dx dy dz . v du dv$$

for $2\phi - w^2$ is the integral of energy and rv that of angular momentum.

If there is no preferential motion, f is a function of w only, so that the latter integral cannot be concerned, and we have

$$N = 2\pi f(V^2 - w^2) . 2w^2 dw . dx dy dz.$$

Consider the frequency-law

$$f(V^2 - w^2) = \begin{cases} (V^2 - w^2)^\lambda & \text{for } w < V \\ = 0 & \text{for } w > V. \end{cases}$$

This law is of the form at which we arrived in the last paragraph, since f falls gradually to zero at a limiting velocity V , and for small values of w approximates to the error law.

The density ρ is obtained by integrating over all values of w

Thus
$$\rho = 2\pi \int_0^V (V^2 - w^2)^\lambda . 2w^2 dw$$

* *M.N.*, vol. lxxvi. p. 78.

or setting $w^2/V^2 = t$

$$\begin{aligned}\rho &= 2\pi V^{2\lambda+3} \int_0^1 (1-t)^\lambda t^{\frac{1}{2}} dt \\ &= 2\pi B(\lambda+1, \frac{3}{2}) V^{2\lambda+3}\end{aligned}$$

where B is the Beta function.

$$= 2\pi B(\lambda+1, \frac{3}{2}) (2\phi)^{\lambda+\frac{1}{2}}.$$

If $\lambda = \frac{7}{2}$, we obtain $\rho = C\phi^5$, which is the adiabatic law for $n=5$, approximately followed in the clusters.

More generally if $\lambda = n - \frac{3}{2}$, we obtain the adiabatic distribution for any value of n .

We thus see that a velocity-law of the type required by general considerations leads to the law of density actually observed in a cluster; but we shall not have progressed far in explaining the observed constitution unless we can throw some light on the curious value $\lambda = 3\frac{1}{2}$.

Since $\rho = C\phi^n$, ϕ and ρ vanish together, *i.e.* at the boundary of the cluster. Hence ϕ differs from the potential, which we have previously taken to be zero at infinity, by a constant, except in the one case when the boundary is at infinity, *viz.* $n=5$. When $n < 5$, V is the velocity which would enable the star to reach the boundary of the cluster, but not the velocity of complete escape. Since stars with velocities slightly greater than V would ultimately fall back into the cluster, there seems no particular reason for their absence.

The observed value $\lambda = 3\frac{1}{2}$, $n=5$ may therefore be attributed to the fact that it gives a cluster in which the limiting velocity is the same as the velocity of escape. In any of the other cases there is a limiting velocity which is unexplained and arbitrary; but for $n=5$ the reason for the limit is obvious, since any star which acquired a greater velocity would quit the cluster.

As has already been explained, no equilibrium is possible if $n > 5$.

These considerations seem to explain why the observed distribution conforms more nearly to the value $n=5$ than to any other value of n —the one property which the observations definitely establish. But it is not necessary to confine ourselves to laws of the form $\rho = \phi^n$, since we have rejected the analogy of the theory of gases. There are other laws which possess all the advantages of $\rho = \phi^5$; and in particular they keep the property that the limiting velocity is equal to the velocity of escape. Some account of these will be given later. It may be suggested that there is a much stronger family resemblance between the laws satisfying this condition than between the adiabatic laws for different values of n , so that the results of observation would be satisfied if the choice between them was left arbitrary and differed in different clusters. In that case the close agreement with $n=5$ may perhaps be attributed to

the fact that the other theoretical distributions tried were utterly inappropriate; as though we jumped to the conclusion that the orbits of double stars were circles, because after trying hyperbolas, lemniscates, and cycloids we found that a circle gave a surprisingly good fit. But the following paragraphs seem to show that there need not be much resemblance between the possible distributions which are left.

7. I am afraid that the narrowing down of the theoretical possibilities to the system represented by $n=5$ turns out to be largely illusory when we no longer confine ourselves to the form $\rho = \phi^n$. The properties which we have used are—

(1) The density distribution is such that it can be maintained in a steady state by a suitable distribution of velocities; and, moreover, the velocity distribution is random in direction, *i.e.* without star-streaming.

(2) The limiting velocity V' is equal to the velocity of escape V .

I shall now consider how general a type of system satisfies these conditions. The second condition, however, imposes practically no restriction. If the frequency function f falls to zero between $w=V'$ and $w=V$, the case is obviously a very special one, and there will be a nearly equivalent distribution of density in which f is small but not zero between V' and V , and therefore satisfies condition (2).

Take an arbitrary distribution of density ρ having globular symmetry; then corresponding to any value of ρ there will be a definite value of ϕ . By Jeans's theorem

$$\rho = 4\pi \int_0^V f(2\phi - w^2) w^2 dw$$

where $V^2 = 2\phi$.

For the conditions (1) and (2) to be satisfied, it must be possible to find a function $f(z)$ positive and finite for positive values of z , and zero for $z \leq 0$, and satisfying the above equation.

I shall assume all ordinary conditions of continuity to be satisfied since there is no object in discussing extravagantly artificial systems. Since ϕ decreases continually from the centre to zero at infinity, ρ may be regarded as a single-valued function of ϕ .

We have, remembering that $f(0) = 0$,

$$\rho' = \frac{d\rho}{d\phi} = 8\pi \int_0^V f'(2\phi - w^2) w^2 dw.$$

Set $w^2 = u$, then

$$\begin{aligned} \rho' &= 4\pi \int_0^{2\phi} f'(2\phi - u) u^{\frac{1}{2}} du \\ &= - \left[4\pi f(2\phi - u) u^{\frac{1}{2}} \right]_0^{2\phi} + 2\pi \int_0^{2\phi} f(2\phi - u) u^{-\frac{1}{2}} du. \end{aligned}$$

The integrated part vanishes. Set $2\phi - u = \xi$, then

$$\rho' = 2\pi \int_0^{2\phi} \frac{f(\xi)}{(2\phi - \xi)^{\frac{1}{2}}} d\xi.$$

This is a case of Abel's integral equation, and the solution is

$$\begin{aligned} 2\pi f(z) &= \frac{1}{\pi} \frac{d}{dz} \int_0^z \frac{\rho' d(2\phi)}{(z - 2\phi)^{\frac{1}{2}}} \\ &= \frac{2}{\pi} \frac{d}{dz} \left\{ - \left[(z - 2\phi)^{\frac{1}{2}} \rho' \right]_0^z + \frac{1}{2} \int_0^z (z - 2\phi)^{\frac{1}{2}} \rho'' d(2\phi) \right\} \end{aligned}$$

where $\rho'' = d^2\rho/d\phi^2$.

If the mass is finite, ρ must vanish at infinity at least as fast as ϕ^3 , so that $\rho'(0) = 0$. The integrated part therefore vanishes. Hence, performing the differentiation,

$$f(z) = \frac{1}{4\pi^2} \int_0^z \frac{\rho'' d(2\phi)}{(z - 2\phi)^{\frac{3}{2}}}.$$

By this result we can determine the appropriate frequency function for a steady state when ρ is given. The condition that $f(z)$ should be positive for all positive values of z (up to the limit required) is obviously satisfied when ρ'' is always positive. The condition is sufficient but not necessary, and ρ'' may be zero or negative near the centre of the cluster. At any rate a distribution for which $d^2\rho/d\phi^2$ is always positive and finite can be a steady state, and will possess all the merits which we have yet found for $\rho = \phi^5$.*

An equivalent condition is that $d\rho/d\phi$ decreases as we pass outwards. There can be no doubt that a very wide variety of density distributions would satisfy this condition.

8. In casting about for some other distinguishing feature of the law $\rho = \phi^5$ which might determine its selection as the type for globular clusters, we may note one distinctive property which may possibly be significant. In any cluster in a steady state the kinetic energy is one-half the exhaustion of potential energy; but it is only for $\rho = \phi^5$ that this result holds for every element of volume and not merely for the whole cluster.†

* It might seem that this includes $\rho = \phi^4$, already dismissed; but that distribution is more strictly written $\rho = (\phi - C)^4$ for $\phi > C$ and $\rho = 0$ for $\phi < C$. Hence $\frac{d^2\rho}{d\phi^2}$ is zero over a certain range.

† *M.N.*, vol. lxxvi. p. 525. If we wish to avoid *localising* the potential energy, the result may be stated in the form—the kinetic energy in each element is one-quarter of the potential energy of the element due to the attraction of the rest of the cluster.

With the same notation as before, the kinetic energy T in an element of volume

$$= \frac{1}{2} \int_0^V f(V^2 - w^2) \cdot w^2 \cdot 4\pi w^2 dw.$$

Hence
$$\frac{dT}{d(V^2)} = \frac{1}{2} \int_0^V f'(V^2 - w^2) \cdot 4\pi w^4 dw$$

$$= \frac{3}{4} \int_0^V f(V^2 - w^2) \cdot 4\pi w^2 dw \quad \begin{array}{l} \text{by integration} \\ \text{by parts.} \end{array}$$

$$= \frac{3}{4} \rho.$$

Hence
$$\frac{dT}{d\phi} = \frac{3}{2} \rho.$$

If then for every element $T = \frac{1}{2} \Omega = \frac{1}{4} \rho \phi$, this gives

$$\frac{d(\rho\phi)}{d\phi} = 6\rho,$$

of which the only solution is $\rho = C\phi^5$.

It is not obvious that any special stability will result from this uniform relation between the kinetic and potential energies at every point, but the property is perhaps worth bearing in mind.

9. Another line of explanation, with respect to which I have made a great many calculations, depends on the possibility that the law $\rho = \phi^5$ might make the entropy of the system a maximum, and so represent a normal state according to the principles of statistical mechanics. The result turns out to be negative, and I give here only an outline of the work.

To consider a general law of density, set $r = \tan \theta$, and therefore

$$\cos \theta = \frac{1}{(1+r^2)^{\frac{1}{2}}}.$$

The angle θ will range from 0 at the centre to $\pi/2$ at infinity. The potential ϕ will vanish for $\theta = \pi/2$, the mass of the cluster being finite, and it is clear that ϕ can be expressed as a Fourier Series * in $\cos \theta$, $\cos 2\theta$, $\cos 3\theta$

Taking as many terms of this as required, we convert $\cos 2\theta$, $\cos 3\theta$, etc., into powers of $\cos \theta$, and obtain

$$\phi = a \cos \theta + b_2 \cos^2 \theta + b_3 \cos^3 \theta + b_4 \cos^4 \theta + \dots \quad (1)$$

Unlike the Fourier Series, a power series of the form (1) is

* Many such expansions are possible; not only is the unit of r arbitrary, but ϕ is undefined from $\theta = \pi/2$ to π . It would be possible to obtain an expansion with only the odd multiples of θ , but I have retained the more general expression.

not necessarily convergent, so that we cannot always obtain in this form a *perfect* representation of a given law of density.

In terms of θ , $\nabla^2 = \cos^4 \theta \frac{\partial^2}{\partial \theta^2} + 2 \frac{\cos^4 \theta}{\sin \theta} \frac{\partial}{\partial \theta}$, whence

$$\nabla^2 \cos^n \theta = n(n-1) \cos^{n+2} \theta - n(n+2) \cos^{n+4} \theta.$$

Accordingly

$$4\pi\rho = -2b_2 \cos^4 \theta + (3a - 6b_3) \cos^5 \theta + (8b_2 - 12b_4) \cos^6 \theta \\ + (15b_3 - 20b_5) \cos^7 \theta + \dots \quad (2)$$

the general term being

$$(m-2)\{(m-4)b_{m-4} - (m-3)b_{m-2}\} \cos^m \theta.$$

From (1) it appears that the whole mass of the cluster is equal to a , and hence a must be positive and finite. If $b_2, b_3 \dots$ are small compared with a , we can by a series of successive approximations, substituting from (1) in (2), obtain an expansion of ρ in powers of ϕ , viz.,

$$\rho = \frac{3}{a^4} \phi^5 + c_4 \phi^4 + c_5 \phi^5 + c_6 \phi^6 + \dots$$

where $c_4, c_5, c_6 \dots$ are small compared with $3/a^4$.

This can correspond to a steady state if we take for the frequency function

$$f = k(V^2 - w^2)^{\frac{3}{2}} \{1 + d_4(V^2 - w^2)^{-1} + d_5 + d_6(V^2 - w^2) + \dots\}$$

where $\frac{3}{a^4} = 2\pi k 2^5 B(\frac{9}{2}, \frac{3}{2})$

$$c_4 = 2\pi k 2^4 B(\frac{7}{2}, \frac{3}{2}) d_4, \text{ and so on.}$$

These equations are determined by the relation

$$\int^V f \cdot 4\pi w^2 dw = \rho.$$

When the coefficients $b_2, b_3 \dots$ are not small, very complex relations must be satisfied in order that the density may be everywhere positive; b_2 must in all cases be negative since it predominates at infinity. If b_2 exists alone, the condition at the centre requires that $-b_2 < \frac{1}{2}a$; but this may be modified by the presence of other coefficients. Further conditions are introduced by the necessity that the expression for f should be convergent.

10. If we consider a primitive aggregation of stars distributed in any manner with any velocities, it seems clear from general considerations that the system will settle down to a state which may be considered steady, since it is only slowly altered by the encounters of the stars. Stars with high velocities

may be lost in the process ; it is highly probable that some will escape since the condition for a steady state, $T = \frac{1}{2}\Omega$, shows that on the average the mean kinetic energy of the stars is as much as a quarter of the kinetic energy required for escape. The system might oscillate at first ; but, since the periods of the individual stars are not isochronous, the oscillations would die out rapidly. The question arises whether the ultimate steady state may not be practically independent of the initial conditions ; and if so, can we show that it is the state which the actual clusters assume ?

We have already found a variety of steady states which could arise from suitable initial conditions ; but it may happen that one of these is overwhelmingly more probable than the others—what is called in statistical mechanics a *normal* state. In the theory of gases it is shown that the most probable state is obtained by making H a minimum, where

$$H = \iiint \iiint f \log f dx dy dz du dv dw.$$

Usually such a state is overwhelmingly more probable than any other. In varying H we must keep constant any quantities which are given for all time in the system, *e.g.* the total mass and energy. As is well known, this criterion is equivalent to that of maximum entropy.

It is not very clear how the H -theorem should be applied to a cluster, since stars have probably escaped, if not in the actual cluster, at least in the system of imaginary clusters contemplated in establishing the theorem. The escaped stars can be neglected in H ; for when the diffusion is so great that not more than one star is contained in any compartment $dx dy dz du dv dw$, however large, no one distribution is more probable than any other. But the mass and energy of the reduced cluster are not those of the original system. For the energy, we may perhaps adopt the condition that the energy of the cluster is not greater than that of the original system, since the escaped stars must carry away positive energy. The mass may be regarded as fixed datum, since we seek the most probable distribution subject to the condition that a definite number of stars remain unescaped ; if no such limitation were imposed we should arrive at the final state after all the stars have escaped.

It is probable that this use of the H -theorem is open to criticism, but I have thought it worth while to apply it tentatively. We assume then that the most probable distribution is that which leads to the lowest value of H , subject to the condition that the mass is given and the energy is not greater than a fixed value. The latter condition is equivalent to fixing the energy, since for otherwise similar systems H will decrease as Ω decreases, and therefore as the energy increases. I find that with the law $\rho = \phi^n$, H decreases as n increases up to 5. The

law $n=5$ is therefore the most probable of these laws. But there is not a true minimum of H here; $\frac{\partial H}{\partial n}$ is still negative, and it is only because higher values of n are excluded for other reasons that $n=5$ appears most probable.

Adopting the more general form

$$\phi = \cos \theta + b_2 \cos^2 \theta + b_3 \cos^3 \theta + \dots$$

where the squares and products of $b_2, b_3, \dots$ can be neglected, we have shown how to find f , and it will be found that H can be determined by actual integration. The work is laborious, but all the integrals reduce to Beta functions and their derivatives.

I find in fact that

$$\delta H = 3.4438b_2 + 1.5000b_3 + 0.8561b_4 + 0.5514b_5 + 0.3820b_6 \\ + 0.2785b_7 + \dots$$

where δH represents the change of H compared with its value for $\phi = \cos \theta$.

(I have found that the coefficient of b_n

$$= \frac{1}{2} B\left(\frac{n+1}{2}, \frac{3}{2}\right) \left[-\frac{9}{2} + \frac{105}{2} \frac{\partial}{\partial \mu} \left\{ \log B\left(\frac{9}{2} + \mu, \frac{3}{2}\right) - \log B\left(n + \frac{7}{2} + \mu, \frac{3}{2}\right) \right\} \right. \\ \left. + n(n+2) \left(\frac{7}{4} \frac{\partial}{\partial \mu} \left\{ \log B\left(\frac{n+1}{2} + \mu, \frac{3}{2}\right) - \log B\left(\frac{n-1}{2} + \mu, \frac{3}{2}\right) \right\} \right. \right. \right. \\ \left. \left. + \frac{7}{2} \frac{\partial}{\partial \mu} \left\{ \log B\left(n + \frac{7}{2} + \mu, \frac{3}{2}\right) - \log B\left(n + \frac{3}{2} + \mu, \frac{3}{2}\right) \right\} \right) \right]$$

where μ is to be put zero after the differentiation,

$$= \frac{1}{2} B\left(\frac{n+1}{2}, \frac{3}{2}\right) \left[-\frac{9}{2} - \frac{105}{2} \left\{ \frac{2}{2n+5} + \frac{2}{2n+3} + \dots + \frac{2}{9} - \frac{1}{n+4} \right. \right. \\ \left. \left. - \frac{1}{n+3} - \dots - \frac{1}{6} \right\} \right. \\ \left. + n(n+2) \left(\frac{7}{4} \left\{ \frac{2}{n-1} - \frac{2}{n+2} \right\} + \frac{7}{2} \left\{ \frac{2}{2n+5} + \frac{2}{2n+3} - \frac{1}{n+4} - \frac{1}{n+3} \right\} \right) \right]$$

and the above numerical values are calculated from this formula.)

Since in any steady state the whole energy is half the potential energy, it is sufficient to consider the variation of the latter. This is easily calculated from (1) and (2), and we find

$$2\delta\Omega = .800b_2 + .589b_3 + .457b_4 + .368b_5 + .305b_6 + .258b_7 + \dots$$

The mass is independent of $b_2, b_3, \dots$ so that $\delta M = 0$, identically.

We have seen that b_2 must be negative, but the other coefficients can be given any small arbitrary values. Although there is some superficial resemblance in the run of the numerical coefficients in δH and $\delta\Omega$, it is clear that δH can be made

negative subject to $\delta\Omega=0$. We therefore conclude that the law $\rho=\phi^5$ (which is equivalent to $\phi=\cos\theta$) does not give the minimum value of H , and adjacent distributions satisfying all the conditions can be actually written down, which are more probable than it, according to the criterion we have employed.

So far as I can ascertain, the true minimum of H does not occur near the solution $\phi=\cos\theta$. We can continually diminish H until the values of $b_2, b_3, \dots$ become too large for f to be expanded in powers of them.

11. I have confined myself for the most part to clusters in which no star-streaming exists, because star-streaming would probably die out fairly rapidly owing to the encounters. It is, of course, possible that, if the clusters are not of great age, star-streaming may persist. The velocity-law appropriate to any distribution of density is then no longer unique; for instance, I have given an example (*M.N.*, vol. lxxiv. p. 13) of a different velocity-law which would give $\rho=\phi^5$. There are various other modifications of the conditions that might be considered, such as a resisting medium or the effect of the stars differing somewhat in mass. Again, on Lindemann's hypothesis of a short time-scale, giving the life of the star as of the order 10^7 years, the density distribution might represent the scattering of a group of stars.

I hope that this paper has shown that the similarity of the distribution in the different clusters constitutes a very striking problem—a problem which is not solved by any of the more obvious theoretical explanations.

Stars with Large Proper Motions between Declination 65° and the North Pole, determined at the Royal Observatory, Greenwich.

(Communicated by the Astronomer Royal.)

The following list of stars with proper motions greater than $20''$ a century is a continuation of the list of 170 stars given in the Introduction to the Greenwich Catalogue for 1900.0.

About two-thirds of the proper motions have been determined by comparison of the positions given in vol. iii. of the Greenwich Astrographic Catalogue with earlier catalogues, Carrington, Lalande, Oeltzen's Argelander, and the catalogues of the Astronomische Gesellschaft. The modern catalogues, Berlin (C.), and the Christiania Catalogue for 1900 have also been used. The remainder have been found by direct comparison of plates taken at intervals of 16 to 20 years. This applies to the part of the sky between declination 65° and 71° . Plates were not available for about one-seventh of this area, and this comparison is, therefore, not complete. In the part of the sky from 81° dec. to the North Pole the proper motions of practically

all the stars in the *Bonn Durchmusterung* have been obtained by comparison with Carrington. The regions from 71° to 81° and 64° to 65° will be continued as opportunity offers, as well as the missing regions in 65° to 71° , so as to obtain all the stars in the *Bonn Durchmusterung* with considerable proper motions.

Comparison of plates is very readily made in the Duplex Micrometer. Corresponding plates are set to agree closely and with a very small orientation. Images are bisected on one plate, and the co-ordinates of the images estimated in the second microscope. The plates can be set so that these are between 0 and 10 on Astrographic scale where $10 = 3''$.

The list contains only a few stars brighter than 9.0 m., as nearly all these are contained in that given in the Nine Year Catalogue. To this limit of magnitude the list of stars with proper motions greater than $20''$ a century in the Greenwich Astrographic zones is probably nearly complete. It may be estimated that there are still about 40 or 50 between magnitudes 9.0 m. to 9.5 m. to be found.

Three stars have been found with proper motions greater than $100''$ a century, viz.—

B. D. $66^{\circ}, 268$	9.2 m.	154''
67°, 552	9.3 m.	110''
77°, 361	9.2 m.	101''

The large proper motion of B.D. $67^{\circ}, 552$ was previously given in the Christiania catalogue for 1900. The largest proper motion of a star of magnitude 9.5 m. (B.D.) is $58'$ a century, and belongs to the Carrington star, B.D. $81^{\circ}, 465$.

The following table gives the number of stars of different magnitudes with proper motions within certain limits for the part of the sky from dec. $+65^{\circ}$ to the North Pole. It is practically complete to 8.9 m., but is progressively more incomplete from 9.0 m. to 9.5 m. The total number of B.D. stars of 9.5 m. with proper motions greater than $20''$ a century may possibly be twice as great as the number in the table.

Mag. m.	$>100''$	$60''-99''$	$40''-59''$	$30''-39''$	$25''-29''$	$20''-24''$	
7.0	1	2	7	7	7	9	33
7.0-7.9	1	3	8	7	8	6	33
8.0-8.4	...	2	3	6	1	14	26
8.5-8.9	...	1	5	11	8	13	38
9.0-9.2	4	1	3	12	10	12	42
9.3-9.4	1	...	2	8	9	11	31
9.5	...	...	5	4	10	12	31
Total	7	9	33	55	53	77	234

The area covered between the Pole and dec. $+65^{\circ}$ is .047, or approximately $\frac{1}{20}$ of the whole sky.

Stars with centennial proper motions > 20".

B.D. Zone No.		Mag.		R.A. (1900).			Dec. (1900).		Centennial P.M.		Total.
		Vis.	Photog.	h	m	s	°	'	R.A.	Dec.	
		m	m						"	"	"
70,	35	9.5	9.8	0	34	17	+70	29	+ 27	+ 2	27
82,	16	9.5	11.1		35	33	82	45	- 25	- 5	25
66,	58	6.8	7.2		38	40	+66	37	+ 21	- 5	21
73,	39	9.4	9.8		44	56	73	56	+ 27	+ 12	29
79,	22	9.3	9.9		50	2	79	35	+ 20	- 11	23
81,	27	8.6	8.7		54	42	81	34	- 19	- 5	20
82,	38	9.1	9.5	1	22	51	82	20	+ 30	+ 1	30
72,	107	8.7	9.4		46	59	72	24		- 3	19
68,	138	9.2	9.5		50	48	68	32	+ 32	- 4	32
81,	74	9.5	10.6	2	5	21	81	20	+ 12	- 24	27
71,	145	9.1	9.2		21	22	71	44	+ 31	- 15	34
...	...		11.3		50	38	89	26	+ 22	- 13	26
68,	224	9.4	9.8	3	2	44	68	52	+ 20	- 12	23
66,	268	9.2	10.0		21	48	66	25	+ 118	- 99	154
74,	157	9.5	10.8		24	46	74	55	+ 20	- 13	24
72,	181	9.0	10.0		26	20	72	45	+ 21	- 15	26
68,	302	9.5	9.6		53	9	68	31	+ 3	- 21	21
69,	253	9.4	10.1	4	15	44	69	52	+ 14	- 19	24
67,	330	9.5	9.9		18	29	67	38	+ 17	- 20	26
66,	355	9.4	9.6		40	48	67	0	- 25	- 19	31
88,	25	9.5	10.7		59	40	88	5	+ 17	- 19	25
74,	290	8.8	9.0	6	23	4	74	22	- 4	- 20	20
72,	322	7.5	8.0		24	35	72	5	- 3	- 27	27
71,	380	9.4	9.5		52	35	71	21	- 3	- 47	47
Kas. 1333		9.4	10.6	7	14	22	76	19	+ 13	- 28	29
86,	118	9.5	11.0	8	12	44	86	1	- 5	- 22	23
67,	552	9.3	9.5		27	23	67	38	- 110	- 5	110
77,	361	9.2	10.9	9	6	16	77	40	- 101	- 3	101
67,	591	9.1	9.5		22	11	67	2	- 18	- 8	20
70,	578	9.5	10.4		39	41	70	14	0	- 33	33
83,	264	9.5	10.6		35	56	83	25	- 14	- 20	24
65,	737	9.2	9.8		42	0	65	47	- 39	- 19	43
69,	558	8.8	9.1	10	0	45	68	56	- 23	- 10	25
75,	403	9.3	10.5		1	41	75	37	+ 26	+ 32	41
69,	572	9.0	9.8		19	54	69	22	+ 19	- 5	20
68,	620	9.5	10.5		44	54	68	7	- 18	- 26	32
75,	436	9.5	10.1	11	3	25	75	4	- 28	- 3	28

Stars with centennial proper motions—continued.

B. D. Zone No.	Mag. Vis. Photog.	R. A. (1900).			Dec. (1900).	Centennial P.M.		Total.
		m	m	h m s		R. A.	Dec.	
79, 379	9.4	10.1	11	52 10	78 57	- 22	+ 6	23
84, 263	9.5	11.3		59 35	84 0	27	+ 9	28
73, 566	9.3	9.4	12	38 17	73 31	- 32	- 5	32
71, 646	9.5	10.2	13	11 45	70 50	- 20	- 8	22
85, 223	9.2	9.8		20 51	85 14	+ 21	8	23
75, 510	9.5	11.1		31 4	75 31	- 44	- 6	44
75, 511	9.5	10.5		33 14	75 1	- 46	- 1	46
77, 521	9.3	9.8		43 49	77 44	- 19	- 8	21
65, 962	9.1	9.6		45 40	65 42	- 20	+ 3	20
86, 197	9.5	11.3		48 36	86 8	- 23	- 3	23
81, 465	9.5	11.0	14	10 20	81 5	+ 18	- 55	58
75, 551	9.4	10.2	15	2 8	75 39	- 31	- 8	32
85, 254	9.4	10.2		18 51	84 55	- 2	- 20	20
65, 1050	9.5	10.0		20 28	65 33	- 14	- 19	24
77, 584	9.1	9.6		22 51	77 29	- 23	+ 29	37
65, 1092	8.8	9.4		56 43	65 41	+ 17	- 13	21
69, 839	9.3	11.3	16	13 24	69 30	+ 3	- 21	21
75, 606	9.2	9.6		54 53	75 53	+ 10	- 23	25
86, 250	9.5	10.5		58 24	86 4	+ 25	- 5	26
69, 940	9.5	10.0	17	41 46	69 16	+ 13	+ 18	22
70, 950	9.4	10.0		42 40	70 30	- 20	+ 6	21
72, 803	8.2	8.1		43 3	72 27	- 11	+ 32	34
67, 1034	9.2	9.5		44 47	67 20	- 10	- 21	23
78, 612	8.0	8.9		53 6	78 25	- 10	+ 20	22
68, 986	9.5	10.6	18	16 35	68 36	+ 12	+ 41	43
65, 1254	9.4	9.7		15 40	65 42	- 24	- 5	25
81, 623	9.5	11.1		27 42	81 10	+ 4	- 21	22
84, 417	9.5	10.5		36 49	84 8	+ 16	+ 16	23
81, 640	9.3	10.1		45 10	81 16	- 3	+ 31	31
68, 1027	9.1	9.6		49 16	68 45	+ 11	+ 20	23
82, 565	9.3	9.4		58 35	82 35	+ 24	+ 28	37
69, 1043	9.0	9.6	19	20 52	69 43	+ 6	+ 22	23
76, 735	9.4	10.2		25 35	76 55	+ 15	+ 36	39
70, 1112	9.2	9.5	20	14 44	70 32	+ 3	- 25	25
83, 582	9.4	10.4		31 44	83 40	+ 16	+ 29	33
76, 810	9.5	10.5		40 19	76 53	- 3	- 22	22
68, 1155	8.3	8.7		44 53	68 11	+ 15	+ 18	23
65, 1512	9.5	10.9		50 44	65 54	+ 24	+ 30	38
65, 1572	9.5	11.2	21	16 33	65 48	+ 17	+ 20	26

Stars with centennial proper motions—continued.

B.D. Zone No.	Mag.		R.A.			Dec.		Centennial P.M.		Total.
	Vis.	Photog.	(1900).			(1900).		R.A.	Dec.	
°	m	m	h	m	s	°	'	"	"	"
73, 943	9.4	10.1	21	42	25	73	29	+ 10	+ 23	25
66, 1454	9.2	...	51	45		66	15	+ 25	+ 13	28
85, 368	9.5	$\left\{ \begin{smallmatrix} 10.9 \\ 11.2 \end{smallmatrix} \right\}$	52	17		85	27	+ 26	+ 8	27
85, 379	9.5	11.4	22	5	19	85	37	- 2	- 23	23
70, 1218	9.5	9.8	7	12		70	49	+ 22	- 7	23
64, 1694	R.Z. Cephei		35	43		64	20	+ 7	+ 17	18
77, 876	9.5	10.4	46	55		77	25	+ 22	+ 7	23
85, 404	9.4	10.5	23	35	56	86	7	+ 18	+ 8	20
82, 731	9.4	10.2	36	29		82	18	+ 23	- 10	25
68, 1405	9.4	9.7	47	30		68	57	+ 25	- 9	27

The stars B.D. 78°, 17; 78°, 601; 74°, 985; 74°, 1018 and 76°, 914 should be omitted from the list, pp. 35-38 of the Introduction to the Greenwich Second Nine Year Catalogue, as the proper motions are not verified. They depended on a comparison with the A. G. Catalogues alone, with only a small difference of epoch.

The Galaxy of the B-Stars. By C. V. L. Charlier,
Assoc. R.A.S.

In the course of the last year I have made a thorough examination of the distribution and the motion of the stars of the spectral type B. The results obtained are in some cases of a general character, and supposing they will interest the members of the R.A.S., I give here a short *résumé* of the memoir, which will appear in the next *Meddelande* of the observatory in Lund.

In the statistical investigations hitherto made the proper motion of a star of, say, the 6th magnitude is directly compared with the proper motion of a star of the first magnitude, though the latter corresponds to a *linear* motion ten times as great as the linear motion of the former star, if both stars are supposed to possess the same luminosity and apparent motion.

One consequence of this treatment is that the resulting mean values of the distances and motions of the stars are subject to great mean errors, and, what is more serious, sometimes even to considerable systematic errors.

Let us consider for a moment a group of stars, all having the same absolute luminosity. The distance r of each indi-

vidual star is determined by the apparent magnitude m , and a single parameter, which we will call R . We have

$$r = R \cdot 10^{0.2m}.$$

The parameter R is equal to the distance of a star of the apparent magnitude 0.0.

The value of R is evidently not the same for all stars. It varies, however, much less than r . It is found, indeed, that with a few exceptions, R has, for all stars, a value smaller than 10 sirimeters, whereas r may amount to 300 sir. or more. It would hence be appropriate to introduce R as a parameter instead of r , even if no regard were paid to the different luminosities of the stars.

The luminosities of the stars generally increase with the temperature. There is, however, probably a relatively large dispersion in the luminosities of the less hot stars (types K and M), whereas the dispersion in the luminosities of the hotter stars is rather small (compare *Meddelande*, No. 68). Especially it is to be expected that the stars of the spectral type B all have nearly the same luminosity. For this reason I have chosen these stars as the subject of this investigation. This choice is all the more appropriate, as the great majority of these stars in the heavens are registered in the admirable *Annals of the Harvard Observatory*. The luminosity of these stars is, indeed, so great that a star of this type, situated—as far as can be concluded—at the limits of our stellar universe, is scarcely fainter than the 8th magnitude. We are thus in the position to get, with the help of the B-stars, what might appropriately be called a skeleton image of the Milky Way.

Taking all stars of the spectral type B together, and using the known values of their proper motions and radial velocities, I obtained from stars brighter than the 5th magnitude a value of the parameter R equal to 4.76 sirimeters. It was, however, found that this value was correlated to the apparent magnitude of the stars, the fainter stars giving a somewhat smaller value, the brighter stars a higher value of R .

This anomaly was found to be due to the difference in luminosity characterising the subclasses B₀, B₁, B₂, B₃, and B₅ belonging to the spectral type B, and used in the Harvard classification of spectra. Leaving smaller divergencies out of the question, the stars of the subclasses B₁ and B₂, which possess the greatest luminosity, have a value of R equal to 7.4 sir., whereas the three remaining subclasses B₀, B₃, and B₅ (B₈ and B₉ are not here considered) are characterised by a value of R equal to 3.3 sir. We get for each subclass, substantially, the same value, whether we determine R from fainter or brighter stars (some 10 stars of the subclass B₀ make an exception).

The value of R being determined, the position in space of each individual star of the spectral type B is directly deduced. It is found that the B-stars form one well-defined cluster, which

from a relatively dense centre gradually grows thinner till a distance of some 150 to 200 sirimometers is reached, when no more stars are to be found.

The centre of this cluster, which may be assumed to coincide with the centre of our solar universe, is situated in a direction $\alpha = 7^{\text{h}}.7$, $\delta = -55^{\circ}.6$ in the constellation *Carina*. It lies in a part of the heaven which is described by Sir John Herschel as one of the most beautiful in the sky. The distance of the centre from the sun amounts to 18.2 sir. The sun lies 4 sir. above (north of) the XY-plane through the centre (defined below).

The distribution of the B-stars follows a rather simple law. The frequency distribution is, indeed, approximately a normal distribution of type A. The dispersion in the principal plane (the XY-plane) amounts to 37.2 sir., and is nearly the same in all directions of this plane. Perpendicularly to it the dispersion amounts to 13.1 sir. The stellar universe (at least that of the B-stars) has an extension nearly three times as great in the plane of the Milky Way as perpendicularly to it.

The density of the B-stars amounts to 0.0026 stars per cub. sir. in the centre. The limit may be set down as some 150 sir. in the XY plane, and to some 50 sir. in the plane perpendicular to it.

The pole of the galaxy, as far as it is determined by the B-stars, lies at

$$(1) \alpha = 12^{\text{h}}.29, \quad \delta = +28^{\circ}.74,$$

not far from the place universally adopted as the pole of the Milky Way.

A galactic system of co-ordinates is introduced, having the Z-axis directed against the point (1). The X-axis is the projection of the direction from the centre to the sun on the galactic XY-plane. Its spherical co-ordinates (as seen from the sun) are

$$\alpha = 20^{\text{h}}.75, \quad \delta = +47^{\circ}.59,$$

a point in the constellation *Cygnus*.

The number of B-stars, registered in the *Annals of the Harvard Observatory*, is 807. The subclasses B8 and B9 are then excluded, and so also are the variables and stars having peculiar spectra. A catalogue of these 807 stars is given in the Memoir, containing the galactic co-ordinates of the stars, and also all information that has been collected regarding them, such as spectral class, apparent magnitude, parallax, etc. The parallaxes of a few stars of type B have been determined through direct observations—most of them by *Gill*—and are in full agreement with those given in the Memoir.

Diagrams are given in the Memoir which show the distribution of the stars projected on planes at right angles.

The discussion of the *velocities* of the stars of the spectral type B confirms the results found by Gyllenberg (*Meldelände*,

Ser. ii., No. 13). The velocity surface is an ellipsoid of revolution, having the short axis equal to 1.3 siriometers per stellar year [a stellar (or galactic) year = 10^6 tropical years] directed towards the pole of the Milky Way. The long axis amounts to 1.9 sir. per stellar year. There is no preferential direction in the movements of these stars in the plane of the Milky Way, such as has been found for the stars of the other spectral types.

The direction of the *vertex* is approximately (though by no means exactly) perpendicular to the radius vector from the centre. It is superfluous to accentuate the importance of this fact for the interpretation of the singularities found in the movements of the stars.

Observations of the Double Star Krueger 60. (Burnham Gen. Cat. 11761, 1880.0 α 22^h 23^m 43^s, δ + 57° 6'.)

By E. E. Barnard. (Plates 9, 10.)

Attention was first called to this star by Professor A. Krueger in his meridian observations, where it was recorded as a wide double with a distance of 12", being No. 60 of a list of double stars which Krueger had noted with the meridian circle. When measuring Krueger 60 in 1890 Professor Burnham found that the preceding component was an unequal double, with a distance of 2".3. Through courtesy, therefore, this double star is called Krueger 60. The true double, however, was discovered by Professor Burnham. Krueger's two stars have no physical connection whatever, and their proximity was purely accidental—the following component being left behind for ever.

No further observations were made of these stars until 1898, when Professor Eric Doolittle found that the small double measured by Burnham in 1890 had a proper motion of nearly 1" annually and that the two were in orbital motion. Since then this insignificant star, which under some proposed conditions of discrimination among double stars might run a chance of being excluded from the regular double star lists because of being too wide and too small to be of interest, has advanced to the front rank in importance.

The main interest about the star is in the fact that it is a binary system in comparatively rapid motion, and that it is relatively near to us.

In *Monthly Notices*, vol. lxviii. p. 629, 1908, the present writer has given a paper on the parallax and proper motion of Krueger 60 in which, from visual observations, the parallax is

$$\pi = +0''.249 \pm 0''.010.$$

The values of the parallax of this star, derived photographically by

EPOCH 1915.7

South



G

H



A B

F

D

C

E



FIG. 1.—KRUEGER 60 AND COMPARISON STARS.

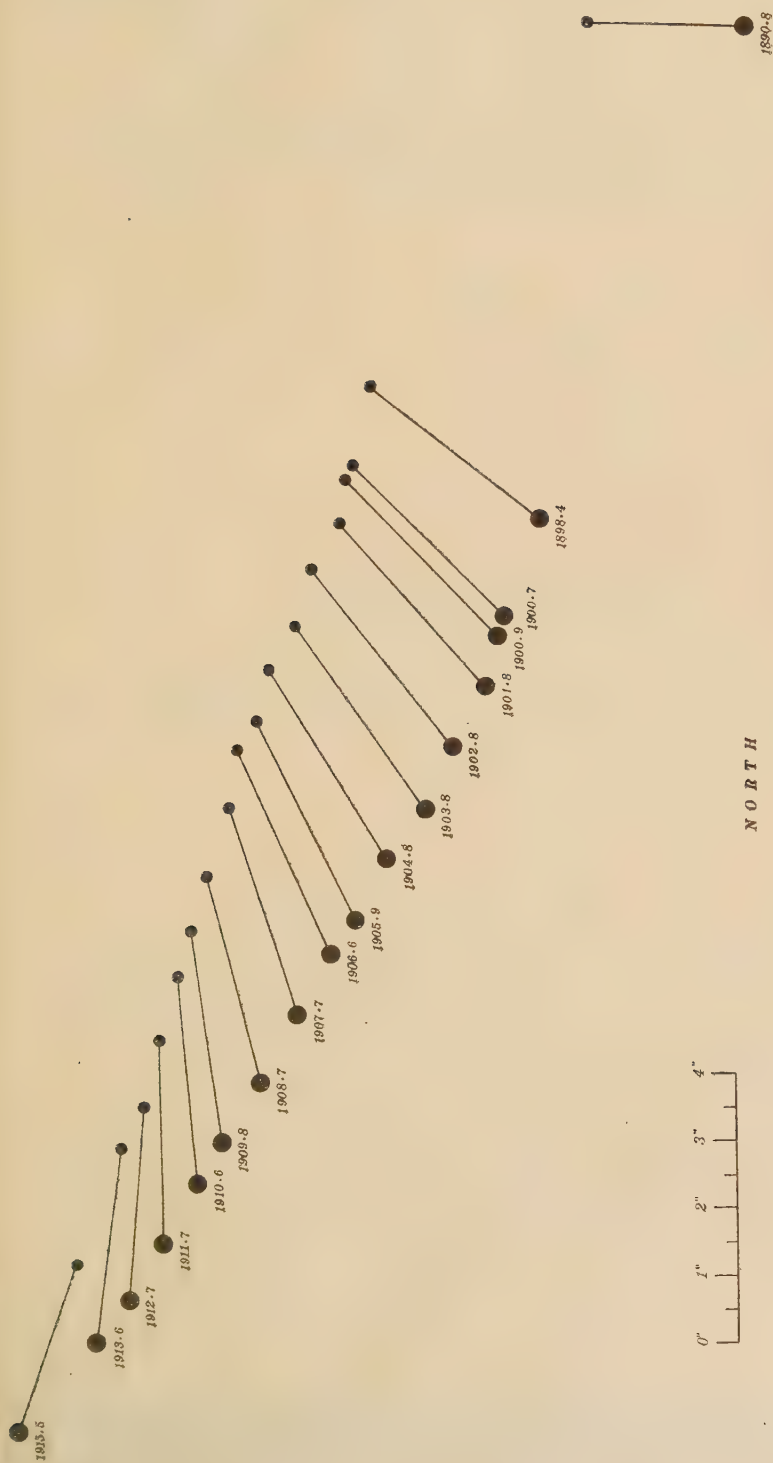


FIG. 2.—KRUEGER 60. THE RELATIVE POSITIONS OF AB WITH RESPECT TO C FROM 1890 TO 1915. (C IS OMITTED.)

Dr. Schlesinger and Dr. Russell independently, are closely consistent with the above value:—

Schlesinger . . . $\pi = +0''.248 \pm 0''.009$.

Russell $\pi = +0''.258 \pm 0''.013$.

From these results it will be seen that, though the star is comparatively faint, it is one of our nearest neighbours in space.

In the paper in *Monthly Notices* already referred to, where a diagram is given at page 642, attention was called to the fact that

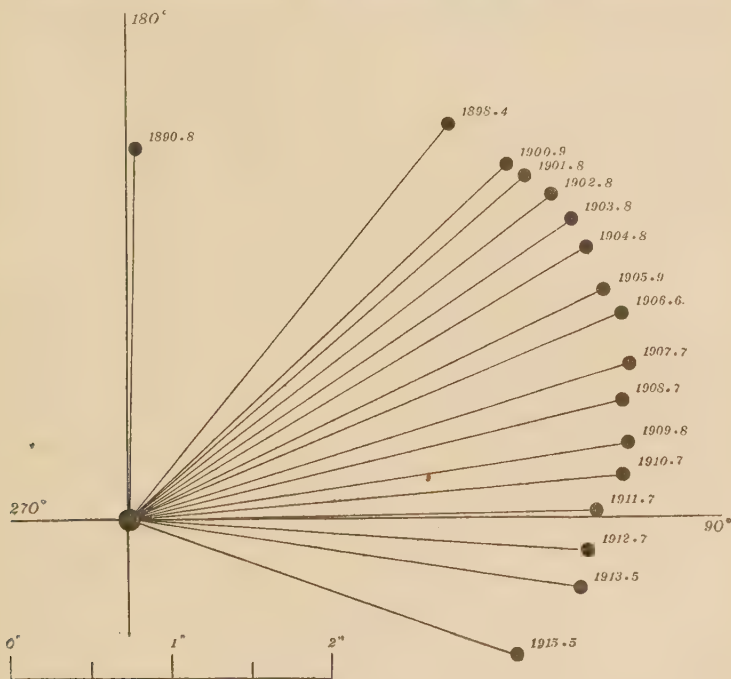


Fig. 3—Krueger 60, AB, from 1890 to 1915.

the motion of A was not rectilinear. I have constructed a new diagram (Plate 10) containing the measures since Professor Burnham's in 1890 to the present time.

The measures of 1890 and 1907 give a motion in the direction $245^{\circ}.0$. Those of 1907 and 1915 give $235^{\circ}.7$. If we take the entire interval from 1890 to 1915, the motion is in $242^{\circ}.1$, which is not in bad agreement with the early value. An inspection of the new diagram shows that the two stars of this system, though quite unequal in brightness—differing by over a magnitude—are nearly equal in mass. One of them, therefore, is intrinsically much less bright than the other. Indeed, Russell found from observations up to 1910 that the mass of B is slightly greater than

that of A, the ratio $\frac{B}{A}$ being 1.14 ± 0.14 .* In plotting the position of B on this diagram it was noticed that the path of A was better represented than that of B with respect to A. I have constructed a new diagram for AB alone, and in this the path of B seems to be affected by a wave-like oscillation, with a very small amplitude, in a period of about six years. To see if any further light could be thrown on this question, the individual measures of AB were plotted, but this does not seem to improve matters. The quantity is really too small (of the order of $\frac{1}{2}$ of a tenth of a second) to mean anything, and is probably due to unavoidable errors in the observations.

To determine independently the masses of A and B, I have lately, at the suggestion of Dr. Schlesinger, begun measures of the two stars separately with respect to the star C.

Recent measures of CE, however, seem to show a slight possible motion in C (or E). Following are the measures:—

CE.

1907.708	129°.65	45".91 (3 n)
1915.859	129°.46	46".43 (4 n)

Dr. Schlesinger has kindly called my attention to a peculiarity in the measured position angles of AB. Where a number of measures at widely different times of the year have been made, the angles in the latter part of the year seem to be as large as those in the first part, though they should be smaller. He suggests that this may be in some way connected with the hour angle. In my measures of this kind, when the stars are bisected for angle the line between them is placed vertical to the line between the eyes, even if the head has to be turned at a large angle to do it. In all the observations of AB, therefore, in east-hour angles the small star is uppermost, while in all measures in west-hour angles it is the lower of the two. The peculiarity is perhaps a physiological effect due to the fact that there is a considerable difference of magnitude between the stars. That it is not due to any instrumental effect is shown by its absence in the measures of A and C, which two stars are nearly of the same brightness. I hope to test the matter by measures on the same night in east- and west-hour angles, and also photographically. For an investigation of the cause of this peculiarity all the hour angles for the time of the measures, both for the present observations and for those printed in *Monthly Notices*, lxviii., are collected in the present paper. Those for the previously printed measures are given in a separate table, while those belonging to the present observations are printed with the measures.

Following are all the measures of Krueger 60 I have obtained since those printed in *Monthly Notices*, vol. lxviii, No. 9, p. 629. An additional column for the hour angle has been inserted in the case of AB, which, of course, will also serve for AC.

* *Ap. J.*, vol. xxxii. p. 363, 1910.

				AB.		
				P.	s.	H.A.
				°	"	h m
1909	'354	May	9	100°43	3°06	E 4 30
	'373		16	99°57	3°16	E 4 10
	'378		18	99°79	3°20	E 4 45
	'392		23	98°74	3°04	E 3 15
	'411		30	98°43	3°02	E 3 50
	'417	June	1	98°82	3°19	E 5 15
	'513	July	6	97°98	3°12	E 0 55
	'584	Aug.	1	98°95	3°17	E 1 0
	'660		29	99°44	3°26	E 1 20
	'814	Oct.	24	97°25	3°11	E 2 10
	'819		26	99°29	3°21	E 1 5
	'838	Nov.	2	97°13	3°14	E 1 50
	'858		9	98°48	3°17	W 3 20
	'871		14	96°78	3°15	E 1 0
	'910		28	100°21	3°06	W 1 5
	'915		30	99°11	3°13	E 0 20
1909	'632			98°78	3°14	
1910	'537	July	15	95°49	3°15	E 1 50
	'542		17	94°47	3°18	E 2 30
	'561		24	95°33	3°11	E 2 20
	'567		26	95°68	3°04	E 4 20
	'580		31	94°49	3°20	E 4 25
	'586	Aug.	2	95°28	3°14	E 1 10
	'605		9	93°52	3°14	E 2 20
	'619		14	93°50	3°06	E 2 5
	'624		16	96°41	3°17	W 0 45
	'693	Sept.	10	94°90	3°13	E 3 30
	'734		25	94°36	3°04	E 0 20
	'753	Oct.	2	93°12	3°28	E 1 10
	'767		7	96°04	2°99	W 0 35
	'772		9	96°61	3°01	W 1 0
	'791		16	93°77	3°10	W 0 20
	'797		18	96°32	3°03	W 1 45
	'810		23	93°48	3°16	W 0 45
	'829		30	94°14	3°04	W 0 35
	'912	Nov.	29	94°99	3°01	W 3 15
	'931	Dec.	6	94°09	3°13	W 2 30
	'944		11	95°55	2°99	W 2 15
1910	'712			94°84	3°10	

{ Single distance,
one setting.

{ Seeing = 3, meas.
good.

Fairly good.

{ Seeing v. bad.
Very difficult.

{ Single distance,
one setting.

{ Seeing = 3, meas.
good.

Fairly good.

{ Seeing v. bad.
Very difficult.

AB.—continued.

		P.	s.	H.A.	
		°	"	h m	
1911	'577 July 30	92°23	3'10	W 0 35	
	'591 Aug. 4	91°44	3'05	W 3 0	
	'621 15	90°93	2'96	W 2 35	
	'659 29	91°05	3'04	W 2 0	
	'673 Sept. 3	90°15	2'91	W 1 35	
	'717 19	91°59	2'98	W 2 45	
	'755 Oct. 3	90°75	2'99	W 1 15	
	'812 24	89°49	3'01	E 1 0	
	'821 27	91°37	2'89	W 4 40	
	'826 29	88°49	2'98	E 1 25	
1911	'705	90°75	2'99		
1912	'554 July 21	87°04	2'85	E 0 35	
	'612 Aug. 11	87°09	2'92	W 0 25	
	'636 20	86°39	2'84	E 2 0	{ Stars dim in thick sky. Seeing fair.
	'650 25	86°81	2'89	W 0 40	{ Seeing fair, but stars faint.
	'669 Sept. 1	85°42	2'80	W 1 25	
	'694 10	86°30	2'94	W 0 35	
	'746 29	87°00	2'90	W 1 35	
	'751 Oct. 1	85°94	2'80	W 0 30	
	'790 15	83°83	2'83	E 2 45	{ Image breaking badly.
	'820 26	84°97	2'99	E 2 0	Seeing very bad.
	'823 27	84°73	2'79	W 2 0	' , , , ,
1912	'704	85°96	2'87		
1913	'499 July 1	80°66	2'92	E 0 20	
	'504 3	82°55	2'74	E 1 40	
	'518 8	80°74	...	E 0 20	Clouds.
	'524 10	82°18	2'89	E 1 15	
	'551 20	80°23	2'76	E 0 40	Seeing fair.
	'567 26	82°17	2'84	E 0 15	Seeing good.
	'589 Aug. 3	81°94	2'81	E 1 0	{ Difficult, seeing very bad.
	'633 19	79°48	2'96	E 2 15	Seeing very bad.
1913	'548	81°24	2'85		

AB.—continued.

		P.	s.	H.A.	
		°	"	h m	
1915	391 May 23	72°11	2'49	E 2 45	Seeing = 2.
	418 June 2	71°54	2'55	E 2 35	
	454 15	69°00	[2'70]	E 2 5	{ Single distances, Omit distances.
	473 22	69°71	2'55	E 1 30	
	492 29	71°82	2'51	E 2 5	
	517 July 8	69°68	2'57	E 1 35	
	550 20	69°42	2'56	E 1 45	
	607 Aug. 10	70°49	2'57	E 2 20	
	741 Sept. 28	68°60	2'56	E 1 45	
	760 Oct. 5	69°09	2'57	W 2 15	
	780 12	69°72	2'57	W 1 50	
	785 14	67°54	2'59	W 5 30	
	796 18	68°19	2'63	W 3 0	
	818 26	67°64	2'48	W 3 40	
	837 Nov. 2	66°93	2'57	W 4 0	
	856 9	67°04	2'60	W 3 35	
1915	917 Dec. 1	67°64	2'49	W 4 55	
1916	021 Jan. 8	68°12	2'26	W 3 10	{ Small star faint in haze.
1915	678	69°12	2'54		

The above measures show beautifully the orbital motion of AB. As the distance is rather rapidly decreasing, the star may soon become a difficult object.

Estimates of the magnitudes of A and B were made on two nights:

1915 Dec. 1	A = 9 ^m ·6	B = 12 ^m .
1916 Jan. 8	A = 9·7	B = 12·5.

AC.

		°	"
1909	354 May 9	59°18	44'44
	373 16	58°80	44'47
	378 18	59°24	44'32
	392 23	58°70	44'70
	411 30	59°14	44'49
	417 June 1	59°28	44'39*

* The mean of two sets of measures made on this date.

AC—continued.

1909'513 July 6	59°50	44°04	.
'581 31	59°55	44°51	
'584 Aug. 1	59°43	44°73	
'660 29	59°56	45°18	
'814 Oct. 24	59°46	45°22	
'819 26	59°54	45°13*	
'838 Nov. 2	59°60	45°01	
'858 9	59°56	45°30	
'871 14	59°50	45°35	
'910 28	59°62	45°14	
'915 30	59°23	45°17	
1909'629	59°35	44°80	

1910'537 July 15	59°52	45°35	
'542 17	59°72	45°53	
'561 24	59°38	45°61	
'567 26	59°36	45°49	
'580 31	59°31	45°79*	
'586 Aug. 2	59°20	45°45	
'605 9	59°50	45°83	
'619 14	59°42	45°62	
'624 16	59°51	45°63	
'693 Sept. 10	59°46	45°76	Seeing excessively bad.
'734 25	59°64	45°90	Seeing poor.
'753 Oct. 2	59°00	46°04	
'767 7	59°45	46°05	Seeing bad.
'772 9	59°49	45°91	
'791 16	59°39	46°26	Seeing very bad.
'797 18	59°46	45°62	Seeing very poor.
'810 23	59°68	46°09	
'829 30	59°46	46°07	
'912 Nov. 29	59°41	46°32	
'931 Dec. 6	59°30	46°26	
'944 11	59°26	46°46	
1910'712	59°42	45°86	

* The mean of two sets of measures made on this date.

AC—continued.

1911'577 July 30	59°55'	46°48'	
'591 Aug. 4	59°54'	46°47'	
'621 15	59°30'	46°56'	
'634 20	59°59'	46°52'	
'659 29	59°43'	46°69'	
'667 Sept. 1	59°61'	46°72'	
'673 3	59°55'	46°51'	
'717 19	59°66'	46°78'	
'755 Oct. 3	59°55'	46°82'	
'812 24	59°38'	46°81'	
'821 27	59°67'	47°06'	
'826 29	59°47'	46°89'	
1911'696	59°52'	46°69'	
1912'554 July 21	59°30'	47°23'	
'612 Aug. 11	59°47'	47°38'	
'636 20	59°51'	47°59'	
'650 25	59°72'	47°58'	
'669 Sept. 1	59°81'	47°52'	
'694 10	59°32'	47°72'	
'746 29	59°51'	47°72'	
'751 Oct. 1	59°55'	47°76'	{ Seeing poor, but measures good.
'790 15	59°25'	47°97'	Seeing very bad.
'820 26	59°28'	47°93'	„ „
'823 27	59°52'	47°85'	„ „
1912'704	59°48'	47°66'	
1913'499 July 1	59°07'	48°13'	Seeing very bad.
'504 3	59°39'	48°13'	
'524 10	59°23'	48°15'	„ „
'551 20	59°41'	48°20'	Seeing bad.
'567 26	59°53'	48°27'	Seeing good.
'589 Aug. 3	59°60'	48°44'	Seeing very bad.
'633 19	59°32'	48°48'	„ „
1913'552	59°36'	48°26'	

AC—continued.

1915'391 May 23	58°82	49°91	
418 June 2	59°03	49°93	
443 11	59°04	49°79	Single distances. Clouds.
454 15	58°99	49°97	
473 22	59°12	50°02	
492 29	59°19	50°02	
517 July 8	59°25	50°06	
550 20	59°18	50°02	
607 Aug. 10	59°33	50°14	
741 Sept. 28	59°22	50°47	
760 Oct. 5	59°35	50°51	
780 12	59°38	50°56	
785 14	59°31	50°64	
796 18	59°24	50°56	
818 26	59°27	50°59	
837 Nov. 2	59°21	50°59	
856 9	59°19	50°89	
1915'917 Dec. 1	59°26	50°74	
1916'021 Jan. 8	59°18	51°11	
1915'666	59°19	50°34	

The following measures of BC have been made for an independent determination of the mass of B. They will be continued.

BC.

1915'818 Oct. 26	58°77	48°03	
837 Nov. 2	58°77	48°02	
856 9	58°52	48°37	Difficult. Seeing very bad.
1915'917 Dec. 1	58°85	48°17	
1916'021 Jan. 8	58°83	48°41	
1915'890	58°75	48°20	

Following is a continuation of the previous measures of A and C with respect to the small stars near:—

AD.

1911'673 Sept. 3	33°66	29°35	
1912'790 Oct. 15	33°47	30°25	D = 14 mag.

A E.

1909'584 Aug. 1	95°06	74°18
'915 Nov. 30	94°74	74°74
1909'750	94°90	74°46

1910'624 Aug. 16	94°93	74°97
'944 Dec. 11	94°40	75°74
1910'784	94°66	75°35

1912'694 Sept. 10	93°65	77°04
'724 21	93°78	76°71
'746 29	93°94	76°81
'751 Oct. 1	93°96	76°67
'790 15	93°81	77°10
'823 27	93°93	77°12
1912'755	93°84	76°91

1913'633 Aug. 19	93°57	77°35
'781 Oct. 12	93°48	77°56
1913'707	93°52	77°45

1915'418 June 2	92°61	78°23	Seeing = 2-3.
'555 July 22	92°89	78°50	
'569 27	92°50	78°71	
'607 Aug. 10	92°87	78°60	
'626 17	92°75	78°61	
'632 19	93°00	78°55	E = 12.9 mag.
1915'568	92°77	78°53	

A F.

1909'915 Nov. 30	283°14	33°57
1910'693 Sept. 10	283°62	32°94
1911'826 Oct. 29	285°93	32°31

AF—continued.

1912'694 Sept. 10	286°17	31'75
'724 21	285°42	31'19
'746 29	286°59	31'49
'751 Oct. 1	286°45	31'67
'790 15	287°11	31'70
1912'741	286°35	31'56

1915'555 July 22	289°72	30'38	
'569 27	290°58	...	Clouds.
'607 Aug. 10	290°12	30'26	
'626 17	289°72	30'30	
'632 19	290°55	30'42	F = 13·7 mag.
'760 Oct. 5	290°45	29'55	F = 14 mag.
'818 26	290°50	29'61	F = 14 mag.
1915'652	290°23	30'09	

CE.

1912'669 Sept. 1	129°76	46'21	Faint in clouds.
1915'760 Oct. 5	129°00	46'35	
'862 Nov. 11	129°49	46'48	
'897 24	129°84	46'71	Images very big and blurred.
'917 Dec. 1	129°51	46'22	
1915'859	129°46	46'44	

Two additional stars have been observed, to be used in future observations of A when it has receded too far from C.

AG.

1912'694 Sept. 10	190°52	61'06	G = 14·5 mag.
'790 Oct. 15	190°92	61'29	G = 13 mag.
1912'742	190°72	61'17	

AH.

1912'790 Oct. 15	235°98	90'02	H = 13·9 mag.
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The two small double stars in the region of Krueger 60 (*M.N.*, vol. lxviii. pp. 653-4) have been observed again, but there does not seem to be any change.

No. 1.

(B. G. C. 11718 = B. D. + 57° 2525.)

1902·764. Oct. 6	247°04	3"19	mag. 9·7	mag. 12*
1913·781 12	246°86	3'23	9·8	12 1/2
1915·670 Sept. 2	246°12	3'27		
741 28	245°26	3'23	9 1/2	12 1/2
1915·706	245°69	3'25	9 1/2	12 1/2

No. 2.

(1903°0 α 22^h 22^m 52^s·1, δ + 57° 13' 33".)

1907·711 Sept. 17	259°03	1"47	mag. 10·5	mag. 11·2*
1910·830 Oct. 30	260°18	1'38	11·5	12·5
1915·670 Sept. 2	263°00	1'36		
722 21	260°06	1'49	10·5	11
1915·696	261°53	1'42	10·5	11

The following Component of No. 2 and a Faint Star North.

1910·830 Oct. 30	340°73	87"93	mag.	mag. 13·3
1915·670 Sept. 2	340°53	88°06		

In the observations of No. 2 it was noticed that a star in the same field and north of it was also double. We will call this No. 3.

No. 3.

(1903°0 α 22^h 22^m 55^s·4, δ + 57° 16' 39".)

1915·722 Sept. 21	77°18	1"41	mag. 10 1/2	mag. 14
741 28	80°67	...		14
917 Dec. 1	78°09	0·87	12	15

One of these distances is wrong. The star is very difficult.†

No. 2 and No. 3.

1915·722 Sept. 21	8°35	187"63.
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* Omitted by mistake from *M. N.*, vol. lxviii, pp. 653-4.

† An observation on 1916 May 27 shows that the second distance is correct. The observation is 1916·404 May 27—80°76 0"94.

These give $\Delta\alpha = +3^s.35$, $\Delta\delta = +3' 5''.6$, from which the above position of No. 3 was obtained.

Another small double star was run across in this region.

No. 4.

(1910.0 α 22^h 26^m 28^s.12, $\delta +57^\circ 18' 5''.1$.)

			mag.	mag.
1910.830 Oct. 30	194 ^o .48	1 ^{''} .84	13 ^o .2	13 ^o .9
'912 Nov. 29	195 ^o .42	1 ^{''} .85	12 ^o .5	13 ^o .0
1910.871	194 ^o .95	1 ^{''} .84	12 ^o .8	13 ^o .4

The North Component of No. 4 and a Star following.

			mag.	mag.
1910.830 Oct. 30	151 ^o .56	40 ^{''} .71	...	12 ^o .9

This last star was compared with Helsingfors-Gotha A.G.C. 13199 on 1910 Oct. 30:

P.A. 21^o.65. Dist. 128^{''}.44,

from which the position of the double was obtained.

The position of the star C was also obtained on this date with respect to the above Helsingfors-Gotha star:

$$\Delta\alpha = -1^m 6^s.8(4 \text{ tr.}) \quad \Delta\delta = -0' 18''.0(2).$$

The resulting place for C is

1910.0 α 22^h 24^m 51^s.9, $\delta +57^\circ 15' 7''.9$.

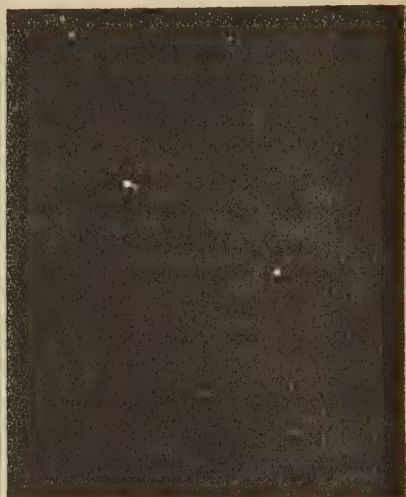
The transits were discordant, so that the $\Delta\alpha$ is given only to the nearest tenth of a second.

The following table contains a collection of the hour angles for all my previous measures of Krueger 60 which were printed in *M.N.*, vol. lxviii. p. 629. It is thought that their importance in connection with the measures will warrant their publication in this paper for the use of future investigators.

Hour Angles for the Earlier Measures.

Date.		H.A.		Date.		H.A.
		h m				h m
1900'937 Dec.	8	W 4 10		1904'560 July	23	W 0 30
'940	9	W 3 35		'570	27	E 2 35
'945	11	W 2 55		'584 Aug.	1	W 1 30
'948	12	W 4 5		'598	6	W 1 0
1901'729 Sept.	23	W 1 0		'642	22	E 0 40
'731	24	W 2 20		'655	27	W 2 10
'748	30	W 2 0		'675 Sept.	3	W 2 5
'751 Oct.	1	W 1 20		'751 Oct.	1	W 1 20
'783	13	W 1 15		'795	17	W 1 0
'805	21	W 1 15		'828	29	W 0 20
'827	29	W 2 0		'905 Nov.	26	W 3 45
1902'744 Sept.	29	W 3 25		1905'904 Nov.	26	W 1 0
'764 Oct.	6	E 0 15		'910	28	W 2 0
'766	7	W 0 55		1906'490 June	28	E 0 30
'785	14	W 1 30		'515 July	7	W 0 20
1903'380 May	19	E 5 20		'523	10	E 3 25
'418 June	2	E 5 10		'561	24	W 1 30
'437	9	E 3 0		'575	29	W 2 5
'454	15	E 2 55		'610 Aug.	11	W 2 10
'473	22	E 2 25		'619	14	W 2 40
'492	29	E 1 10		1907'289 Apr.	16	E 4 40
'495	30	E 1 10		'500 July	2	E 1 40
'511 July	6	E 1 10		'528	12	E 0 15
'514	7	E 1 15		'571	28	E 5 25
'645 Aug.	24	W 1 10		'577	30	E 5 50
'664	31	W 1 45		'596 Aug.	6	E 0 25
'667 Sept.	1	W 0 20		'601	8	E 0 15
'725	22	W 0 45		'610	11	E 5 0
'741	28	W 0 20		'615	13	W 2 35
'818 Oct.	26	W 1 30		'634	20	E 4 45
1904'335 May	2	E 4 20		'706 Sept.	15	W 4 45
'338	3	E 4 0		'749 Oct.	1	W 0 30
'373	16	E 3 45		1908'300 Apr.	19	E 5 0
'393	23	E 4 0		'338 May	3	E 5 20
'434 June	7	E 2 10		'376	17	E 3 40
'502 July	2	E 0 45		'382	19	E 3 25
'521	9	W 0 20		'390	22	E 3 40
'527	11	W 0 45		'401	26	E 3 25

Two photographs of Krueger 60 are given below. These were made by the writer in 1908 and 1915 with the large refractor, with the regular screen and Cramer Instantaneous Iso plates. They are on exactly the same scale, and show the motion of the system AB with respect to C, and also the orbital motion of AB in the interval.



1908 July.

1915 September.

Photographs of Krueger 60, AB and C.
(Both photographs are on the same scale).

Yerkes Observatory, Williams Bay, Wisconsin:
1916 March 6.

Observations of V Cassiopeiae (Ch. 8324) in 1910-16.
By A. N. Brown, M.A.

The 242 observations of the Long Period Variable V Cassiopeiae, discovered by Anderson in 1893, which are given below, are in continuation of those published in *Monthly Notices*, vol. lxx. p. 627. Light curves have been drawn from them, from which it appears that 9 maxima, and 10 minima, as shown in the following table, occurred between 1910 June 3, and 1916 April 14.

Maxima.			Minima.		
Date.	Julian date.	Mag.	Date.	Julian date.	Mag.
1910 October 28	8973	7.6	1910 July 18	8871	12.7
1911 June 11	9199	7.8	1911 March 1	9097	12.3
1912 February 1	9434	8.2	1911 September 30±	9310	12.2
1912 September 6	9652	8.1	1912 June 15	9569	12.2
1913 April 27	9885	8.1	1913 January 4	9772	12.2
1913 November 30	0102	7.8	1913 August 18	9998	12.2
1914 August 2	0347	8.2	1914 April 6	0229	12.2
1915 March 9±	0566	8.2	1914 November 21	0458	12.5
1915 November 1	0803	7.9	1915 July 10	0689	12.2
			1916 February 19±	0913	12.1

All the above phases appear reliable, except the eighth maximum, and the third and the tenth minima, the dates of which are somewhat uncertain owing to paucity of observations.

The shape of the curve in the case of the first, sixth, and ninth of these maxima is pointed. At the seventh it is flat. All the others are rounded, as at the maximum of 1910 March 21, which is shown in the curve published in the number of *Monthly Notices* mentioned above. At minimum light the curve is invariably flat, the rise beginning abruptly, and very rapidly, but there is nearly always a marked pause in the rate of increase of the light before the maximum is reached.

The period derived from my observations since they began is 226.6 days. That given by H. C. O. in the *Second Catalogue of Variable Stars*, published in 1907, is 231.5 days.

The different columns of the records below are arranged as follows. The first gives the date, the second the last four figures of the Julian date. The comparisons in the third column were for the most part made by the "fractional" method, but each is here expressed in steps, the value of a step in every case being .1 magnitude. Sometimes, however, fractional and step estimates were made upon the same occasion. In these cases the different estimates are separated by a semicolon, which is also used to divide different fractional and step estimates from each other. Thus on 1910 Sept. 11 two step estimates are shown; on Sept. 12 two fractional; on Sept. 20 one fractional followed by one step. The two methods are in this way clearly distinguished from each other. All the comparison stars used are numbered as in Hagen's *Atlas*, the negative sign after the comparison star implying that the variable was fainter, the positive sign that it was brighter than the star in question. As regards the magnitude of each comparison star the Harvard values have been employed throughout, those values having been obtained from the column headed H. P. in Series VI. of Hagen's *Atlas Stell. Var.* In the last column the magnitude deduced from the comparisons is given, a mean being taken where two or more separate estimates have been made. In such cases fractional and step estimates are given equal weight.

The great majority of these observations were made with a

5-inch refractor with powers varying from 25 to 200, but at the brighter stages a 3-inch refractor was sometimes used.

The Observations.

Date.	Julian date.	Comparisons.	Deducted Mag.	Date.	Julian date.	Comparisons.	Deducted Mag.
1910				1910			
June 3	8826	55-4, 64+1	11.7	Nov. 11	8987	4-4.5, 5+1.5	8.2
14	8837	69-2, 73+1	12.2	16	8992	5-1	8.4
19	8842	69-2, 73+1	12.2	20	8996	4-5.5, 5+1	8.2
July 3	8856	75-4	12.8	22	8998	5-1.5, 6+1.5; 5-1.5, 7+1.5	8.5
11	8864	75-3	12.7	25	9001	=6; =7	8.6
13	8866	75-3	12.7	28	9004	6-1.5; 7-1	8.7
23	8876	75-3	12.7	Dec. 15	9021	19-1	9.8
Aug. 6	8890	75-3	12.7	19	9025	10-7, 24+2; 17-1, 21+1	9.7
10	8894	75+2	12.2	22	9028	22-0.5, 29+1	10.0
30	8914	22-4, 34+1; 29-2	10.3	24	9030	22-2.5, 29+2.5	10.2
Sept. 1	8916	22-4, 34+1; 25-2, 34+2	10.2	27	9033	34+1.5	10.3
3	8918	22-4, 34+1; =29	10.2	29	9035	22-4, 34+1	10.3
8	8923	22-1	10.0	1911			
11	8926	21-1; 22+1	9.9	Jan. 1	9038	=34	10.4
12	8927	10-8, 24+1; 17-1, 21+1	9.8	6	9043	34-3, 43+1.5	10.7
20	8935	10-7, 24+2; 17+2	9.5	30	9067	64-1.5, 69+0.5	12.0
24	8939	10-6, 24+3; =14	9.5	31	9068	64-1.5, 69+0.5	12.0
27	8942	7-2, 13+7; 10-4.5, 24+4.5	9.3	Feb. 19	9087	69-2.5, 73+1	12.2
29	8944	7-4.5, 11+1; 10-2, 24+6.5	9.1	22	9090	69-2.5, 73+1	12.2
Oct. 6	8951	7-2.5, 16+7; 10-1, 24+8	9.0	25	9093	73+1	12.2
15	8960	4-5, 5+1	8.2	Mar. 2	9098	69-2.5, 73+0.5	12.3
21	8966	4-4, 5+2	8.1	7	9103	69-2.5, 73+1	12.2
28	8973	4+1	7.6	21	9117	69-1, 73+1	12.1
Nov. 1	8977	4-1	7.8	25	9121	64-1.5, 69+1	11.9
4	8980	4-1.5, 5+4.5	7.9	Apr. 3	9130	32-4.5, 43+1.5	10.8
6	8982	4-4.5, 5+1.5	8.2	16	9143	17-1, 21+1	9.7
				19	9146	10-7, 24+2; 17-1, 21-1	9.7
				23	9150	10-5, 24+3.5; 17+1	9.5
				May 2	9159	6-4, 12+2; 7-1.5, 13+5.5; 10+1	8.9

The Observations—continued.

Date.	Julian date.	Comparisons.	Deducted Mag.	Date.	Julian date.	Comparisons.	Deducted Mag.
1911				1912			
May 4	9161	6-4, 12+2;	9.0	Jan. 26	9428	4-4.5, 5+1.5	8.2
		7-3.5, 13+3.5		27	9429	4-4.5, 5+1.5	8.2
8	9165	5-2.5, 7+0.5	8.6	28	9430	4-4.5, 5+1.5	8.2
14	9171	5-2, 6+1	8.5	29	9431	4-4.5, 5+1.5	8.2
22	9179	4-5, 5+0.5	8.2	Feb. 2	9435	4-4.5, 5+1.5	8.2
24	9181	4-4.5, 5+1.5	8.2	4	9437	4-4.5, 5+1.5	8.2
27	9184	4-4.5, 5+1.5	8.2	11	9444	4-5, 5+1	8.2
29	9186	4-4, 5+2	8.1	Mar. 3	9465	=10	9.0
June 1	9189	4-3, 5+3	8.0	6	9468	15-0.5, 17+0.5	9.6
6	9194	4-1, 5+5	7.8	7	9469	15-0.5, 17+0.5	9.6
7	9195	4-1, 5+5	7.8	10	9472	15-0.5, =17	9.6
13	9201	4-1, 5+5	7.8	Apr. 3	9496	34-0.5, 40+2.5	10.5
17	9205	4-1, 5+5	7.8	May 17	9540	51-1, 55+1	11.3
27	9215	4-3.5, 5+2.5	8.1	June 6	9560	69-2, 73+1	12.2
July 2	9220	4-4.5, 5+1.5	8.2	10	9564	69-2, 73+1	12.2
5	9223	4-4.5, 5+1.5	8.2	20	9574	69-2.5, 73+1	12.2
10	9228	4-5, 5+0.5	8.2	23	9577	69-2.5, 73+1	12.2
18	9236	6-1.5; 7-1.5,	8.8	July 13	9597	32-1	10.4
		11+4; 8-0.5,		15	9599	22-4, 34+1;	10.3
		10+1.5				32+0.5	
21	9239	5-2.5, 7+1	8.5	25	9609	11-1.5, 15+3	9.2
25	9243	10-1	9.1	Aug. 29	9644	4-4, 5+2	8.1
30	9248	10-6, 24+3	9.6	30	9645	4-4, 5+2	8.1
31	9249	8-7, 24+3.5;	9.2	Sept. 2	9648	4-4, 5+2	8.1
		13+4, 15+6		17	9663	4-4.5, 5+1.5	8.2
Oct. 4	9314	69-2, 73+0.5	12.2	21	9667	4-4.5, 5+1.5	8.2
28	9338	55-2	11.6	22	9668	4-4.5, 5+1.5	8.2
30	9340	51-0.5, 55+1	11.3	27	9673	4-4.5, 5+1	8.2
Nov. 1	9342	51-1, 55+0.5	11.3	Oct. 2	9678	=5	8.3
13	9354	32-5, 43+1;	10.7	4	9680	5-1.5, 6+1.5	8.5
		34-2.5, 40+1		6	9682	6-1.5, 12+4.5;	8.7
21	9362	32-3, 43+3;	10.6			8+1.5; 7-1.5,	
		34-1.5, 40+1.5				11+3.5	
Dec. 11	9382	10-7, 24+2;	9.6	17	9693	6-5, 12+1;	9.0
		15-1; 19+3				7-4, 11+1;	
15	9386	13-1, 15+1	9.4			=10	
19	9390	10-5.5, 24+4	9.5	Nov. 3	9710	22-4, 34+0.5;	10.3
25	9396	10-6, 24+3	9.6			32+1	

The Observations—continued.

Date.	Julian date.	Comparisons.	Deducted Mag.	Date.	Julian date.	Comparisons.	Deducted Mag.
1912				1913			
Nov. 29	9736	51-0'5	11'3	July 11	9960	55-2	11'6
30	9737	51-1'5, 55+1	11'3	20	9969	=69	12'0
Dec. 12	9749	51-4	11'6	24	9973	69-1, 73+2	12'1
16	9753	69-2	12'2	29	9978	69-2'5, 73+1	12'2
30	9767	69-2'5, 73+1	12'2	Sept. 6	0017	69-2'5, 73+1	12'2
1913				8	0019	69-2'5, 73+1	12'2
Jan. 1	9769	69-2'5, 73+1	12'2	24	0035	55-3, 64+1	11'7
5	9773	69-2'5, 73+1	12'2	28	0039	51-1'5, 55+1	11'3
12	9780	69-2'5, 73+1	12'2	Oct. 3	0044	43-2, 51+1	11'1
26	9794	69-1'5, 73+1'5	12'2	24	0065	22-3, 34+2	10'2
31	9799	69-1, 73+2	12'1	Nov. 27	0099	4-1, 5+5	7'8
Feb. 8	9807	51+1	11'1	Dec. 18	0120	4-4'5, 5+1'5	8'2
22	9821	19-2	9'9	28	0130	4-4'5, 5+1'5	8'2
Mar. 2	9829	13-1, 15+1;		30	0132	4-5, 5+1	8'2
		14+2; 16+2	9'2	1914			
8	9835	6-5, 12+1;	9'1	Jan. 3	0136	=5	8'3
		7-4, 11+0'5		5	0138	5-0'5, 7+2'5	8'4
11	9838	6-4'5, 12+1'5; 9'1		23	0156	15-0'5=17	9'6
		=11		26	0159	10-8, 24+1;	9'8
23	9850	5-2, 7+0'5;=6	8'6			17-1'5, 21+0'5	
30	9857	5-1'5, 6+1'5	8'5	Feb. 2	0166	22-4, 34+1	10'3
May 1	9889	4-4, 5+2	8'1	15	0179	43-2, 51+0'5	11'1
5	9893	4-5, 5+1	8'2	18	0182	43-2, 51+1	11'1
6	9894	4-4'5, 5+1'5	8'2	22	0186	=51	11'2
10	9898	4-4'5, 5+1'5	8'2	24	0188	55-1, 64+3	11'5
13	9901	4-5, 5+1	8'2	Mar. 16	0208	69-2'5, 73+1	12'2
16	9904	4-5, 5+1	8'2	21	0213	69-2'5, 73+1	12'2
25	9913	5-2, 7+1; 6+1	8'5	27	0219	69-2'5, 73+1	12'2
30	9918	7-0'5, 8+1'5	8'7	Apr. 25	0248	69-2'5, 73+1	12'2
June 4	9923	6-4'5, 12+1'5; 9'0		27	0250	69-2, 73+1	12'2
		7-4, 11+1		May 15	0268	=51	11'2
7	9926	12-2, 15+1;	9'3	17	0270	43-1'5, 51+1'5	11'1
		14+2; 16+3;		21	0274	=43	10'9
		17+3		24	0277	42+2	10'6
23	9942	=34	10'4	26	0279	34-2, 40+1	10'6
29	9948	34-2, 40+0'5	10'6	June 4	0288	22-3, 34+2	10'2
July 3	9952	43-2, 51+0'5	11'1	14	0298	15-0'5, =17	9'6

The Observations—continued.

Date.	Julian date.	Comparisons.	Deduced Mag.	Date.	Julian date.	Comparisons.	Deduced Mag.
1914				1915			
June 21	0305	6-4'5, 12+1'5; 9'0 7-4, 11+1; 8-1'5, 10+1		Apr. 23	0611	6-5, 12+1; 9'1 7-4, 11+1	
29	0313	=8	8'8	29	0617	6-5, 12+1; =11	9'1
July 4	0318	5-2, 7+1	8'5	May 7	0625	=34	10'4
8	0322	6+1	8'5	23	0641	43-2, 51+1	11'1
9	0323	5-2, 6+1	8'5	30	0648	52-1	11'4
17	0331	4-4'5, 5+1'5	8'2	June 6	0655	55-2'5, 64+1'5	11'7
29	0343	4-4'5, 5+1'5	8'2	July 11	0690	69-2'5, 73+1	12'2
Aug. 14	0359	4-4'5, 5+1'5	8'2	15	0694	69-2'5, 73+1	12'2
25	0370	5+1	8'2	28	0707	=69	12'0
27	0372	=5	8'3	Sept. 11	0752	6-4, 9+2; 8'9 7-1'5, 11+3'5; 8-1, 10+1	
29	0374	=5	8'3	16	0757	7-2, 11+3	8'8
Sept. 5	0381	7-3, 11+2; 8'9 8-1'5, 10+0'5		19	0760	5-1'5, 7+1'5	8'5
11	0387	6-5, 12+1; =11	9'1	25	0766	4-5, 5+1	8'2
17	0393	=21	9'8	29	0770	4-4'5, 5+1'5	8'2
20	0396	19-1, 22+1	9'8	Oct. 4	0775	4-5, 5+1	8'2
24	0400	22-3, 34+2	10'2	18	0789	4-3, 5+3	8'0
27	0403	22-4, 34+1	10'3	29	0800	4-1'5, 5+4'5	7'9
29	0405	34+1	10'3	Nov. 5	0807	4-1'5, 5+4'5	7'9
Oct. 11	0417	43-1, 51+2	11'1	9	0811	4-4, 5+2	8'1
18	0424	51-1, 55+1	11'3	17	0819	4-4'5, 5+1'5	8'2
Nov. 14	0451	73-2	12'5	26	0828	=7	8'6
18	0455	73-2	12'5	Dec. 6	0838	=15	9'5
Dec. 8	0475	69-2, 73+1	12'2	1916			
16	0483	69-2, 73+1	12'2	Jan. 7	0870	34-1, 43+1	10'8
1915				13	0876	43-1, 51+2	11'0
Jan. 5	0503	=32	10'3	22	0885	51-2; =55	11'4
11	0509	22-4, 34+1	10'3	Feb. 5	0899	69-1, 73+2	12'1
16	0514	19-1'5, 22+0'5	9'9	Apr. 5	0959	43-1'5, 51+1'5	11'1
Feb. 15	0544	10-0'5	9'1	9	0963	32-4'5, 43+1'5; 10'7 =40	
24	0553	4-4'5, 5+1'5	8'2	14	0968	22-3, 34+1'5; 10'1 29+1	
28	0557	4-4'5, 5+1'5	8'2				
Apr. 17	0605	6-4, 12+2; 9'0 7-3, 11+2					

The Comparison Stars used.

Hagen's Number.	H.P. Magnitude.	Hagen's Number.	H.P. Magnitude.	Hagen's Number.	H.P. Magnitude.
3	6.3	14	9.4	34	10.4
4	7.7	15	9.5	40	10.7
5	8.3	16	9.5	42	10.8
6	8.6	17	9.6	43	10.9
7	8.6	19	9.7	51	11.2
8	8.8	21	9.8	55	11.4
9	8.9	22	9.9	64	11.8
10	9.0	24	9.9	69	12.0
11	9.1	25	10.0	73	12.3
12	9.2	29	10.1	75	12.4
13	9.3	32	10.3		

Ludgrove, New Barnet, Herts :
1916 April 17.

The Irregular Variable RS Camelopardalis.

By C. Martin and H. C. Plummer.

1. The variability of this star, which is B.D. +79°·286 or A.G. Kazan 1654, was discovered from photographic plates by Mme. Ceraski * in 1905. The announcement intimated that the period appeared to be short, but no definite period has been found, and the present work proves that the star must be classed as irregular. RS Camelopardalis was under observation at Dunsink from 1909 October to 1914 April, during which time Mr. Martin secured 274 exposures on the region with the 15-inch reflector. In spite of difficulties the continuity of the series was fairly maintained for more than four years, but interruptions having then occurred, it was felt that so large a share of the available observing time could no longer be devoted to a single object. The observations were therefore discontinued. It should be stated that the work was begun under the direction of Professor Whittaker.

2. The magnitudes of the comparison stars were derived from two plates exposed as nearly as possible under similar conditions (a) on the region of the variable, and (b) on the χ Persei cluster. The magnitude scale of these plates is deduced from certain standard stars in the latter region, for which the magnitudes have been determined by Schwarzschild. The results are given in Table I. Here m_1 and m_2 are the magnitudes of the comparison

* *A.N.*, 4047.

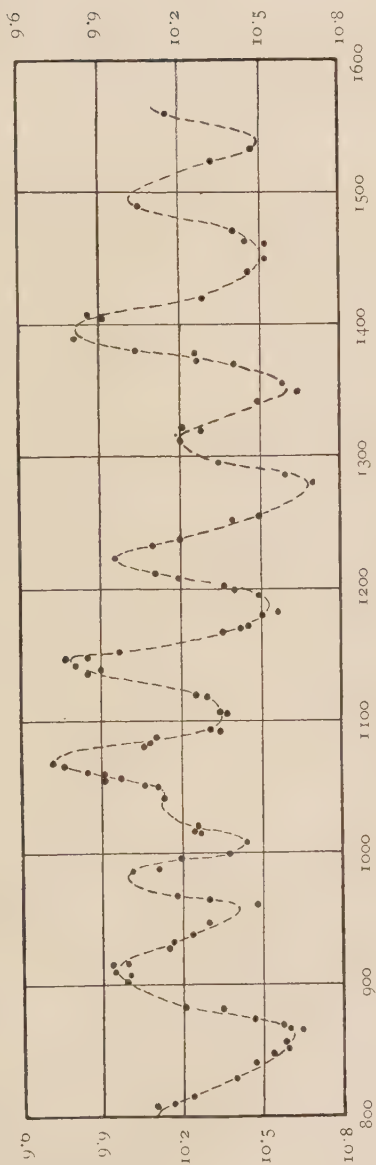
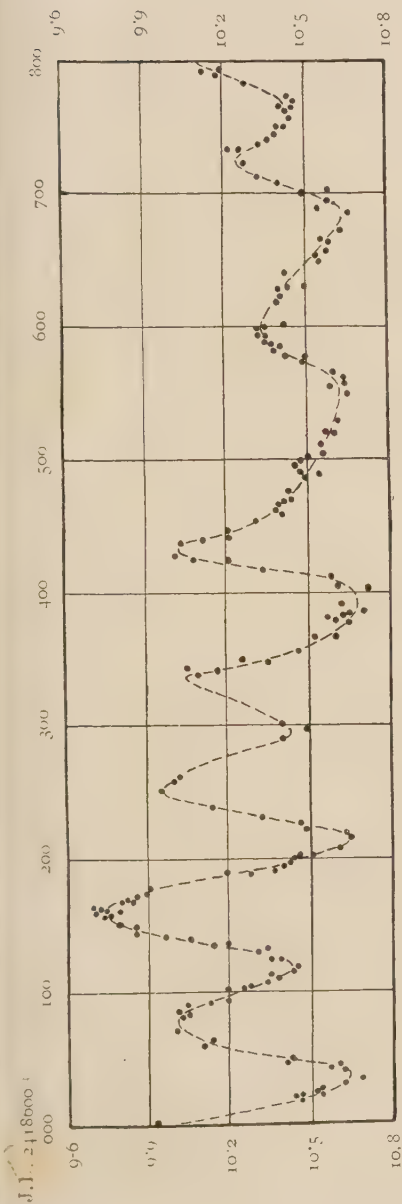


FIG. 1.—PHOTOGRAPHIC LIGHT CURVE OF RS CAMELOPARDALIS.

TABLE I.

Star.	B.D. +79°.	Pl. 219, <i>m</i> ₁ .	Pl. 227, <i>m</i> ₂ .	Mean, <i>m</i> .	Corr.	Final <i>m'</i> .	P.E.	Range.
<i>a</i>	279	9·85	9·94	9·90	+·015	9·91	±·020	·18
<i>b</i>	283	10·24	10·30	10·27	-·013	10·26	±·025	·24
<i>c</i>	287	10·39	10·39	10·39	-·020	10·37	±·026	·23
<i>d</i>	284	10·68	10·71	10·70	-·003	10·70	±·020	·20
<i>e</i>	285	10·94	10·96	10·95	+·013	10·96	±·022	·22

stars deduced from the two plates, and *m* is the corresponding mean. This is adopted in the reduction of the exposures on the variable. A series of residuals is thus obtained from each star, the mean of which is given in the sixth column. By applying these corrections to *m*, the final set of photographic magnitudes *m'* is obtained. The corrections are, however, so small that it is unnecessary to revise the magnitudes of the variable, which are therefore based on the system *m*. In the eighth column are given the probable errors for a single exposure which result from the series of residuals. The extreme range shown by the residuals appears in the last column, and gives satisfactory evidence that none of the comparison stars is sensibly variable.

3. The details of the observations are given in Table II. The date, with G.M.T. for the middle of each exposure, appears in the first column. The same, expressed in Julian Day and decimal fraction, is given in the second column. In the third column is recorded the length of exposure, and the fourth column gives the deduced photographic magnitude of RS Camelopardalis. The general nature of the light variation can only be conveyed in a graphical form. This is shown in fig. 1, in which the magnitudes of the variable are plotted, together with a curve drawn by graphical interpolation.

TABLE II.

Date G.M.T.	Julian Day.	Exposure.	Phot. Magn.	Date G.M.T.	Julian Day.	Exposure.	Phot. Magn.
1909. h m	2418	min.		1909. h m	2418	min.	
Oct. 20 12 21	600·514	35	9·91	Dec. 6 14 0	647·583	40	10·41
Nov. 7 12 29	618·521	35	10·46	8 13 22	649·557	45	10·42
9 12 23	620·516	45	10·44	20 9 59	661·416	45	10·10
10 12 17	621·512	45	10·46	23 11 33	664·481	33	10·13
12 13 3	623·544	45	10·53	31 11 36	672·483	45	10·00
14 13 54	625·580	45	10·52	1910.			
15 12 32	626·522	45	10·53	Jan. 10 9 37	682·401	28	10·02
20 10 15	631·428	45	10·62	11 9 34	683·399	40	10·04
23 11 27	634·477	45	10·71	12 9 47	684·408	60	10·01
29 11 15	640·468	45	10·62	19 9 53	691·412	45	10·04
Dec. 1 9 14	642·385	45	10·57	20 10 36	692·442	40	10·12
3 13 59	644·583	35	10·60	21 9 16	693·386	45	10·18

TABLE II.—continued.

Date G.M.T.				Julian Day.	Exposure.	Phot. Magn.	Date G.M.T.				Julian Day.	Exposure.	Phot. Magn.
1910.	h	m		2418	min.		1910.	h	m		2418	min.	
Jan.	29	10	5	701'421	45	10'18	May	10	10	43	802'448	45	10'45
	31	11	38	703'485	45	10'26		11	10	22	803'432	45	10'51
Feb.	2	9	15	705'385	50	10'27	June	17	11	8	809'463	45	10'61
	3	9	32	706'397	45	10'33		25	10	41	817'446	45	10'65
	8	8	25	711'351	50	10'37		27	11	21	819'472	41	10'64
	10	8	12	713'341	45	10'34		31	11	41	823'487	45	10'48
	14	11	4	717'461	45	10'43		6	11	28	829'478	45	10'46
	15	7	18	718'305	30	10'45		9	11	48	832'492	20	10'32
	21	7	46	724'324	45	10'38		16	11	44	839'490	45	10'14
	22	8	9	725'339	40	10'35		30	12	42	853'530	45	9'95
Mar.	28	8	15	731'343	45	10'30	July	8	11	29	861'479	32	10'00
	1	8	28	732'353	30	10'33		11	11	37	864'484	45	10'02
	4	7	30	735'313	35	10'13	Aug.	9	10	39	893'445	45	10'40
	5	8	4	736'336	35	10'18		15	12	1	899'501	45	10'49
	10	8	0	741'333	40	10'05	Sept.	17	13	8	901'457	45	10'40
	11	8	3	742'335	35	9'96		24	9	59	939'416	45	10'08
	15	8	4	746'336	40	9'85	Oct.	27	9	27	942'394	30	10'16
	18	8	45	749'365	45	9'85		29	8	5	944'337	45	10'04
	21	10	7	752'421	45	9'78	Nov.	3	9	56	948'414	21	10'35
	27	8	49	758'367	35	9'73		5	8	1	950'334	45	10'25
	28	9	33	759'398	35	9'75		11	10	46	956'449	45	10'46
	29	11	23	760'474	35	9'74		21	7	18	966'305	45	10'60
	30	11	13	761'467	35	9'70		22	9	15	967'385	45	10'52
	31	11	3	762'460	35	9'78		2	12	3	978'502	45	10'65
Apr.	1	11	31	763'479	50	9'72		4	11	36	980'483	45	10'60
	2	13	14	764'552	35	9'69		7	7	33	983'315	45	10'57
	7	9	0	769'375	40	9'80		8	7	22	984'307	36	10'63
	8	13	14	770'552	45	9'82		9	10	47	985'450	45	10'64
	9	11	10	771'465	45	9'83		11	9	42	987'405	35	10'70
	10	12	49	772'534	45	9'85		16	12	18	992'513	30	10'62
	13	12	37	775'526	45	9'88					2419		
	16	14	3	778'585	45	9'90	Dec.	28	6	51	004'286	45	10'72
	26	12	39	788'528	45	10'27		29	9	43	005'406	45	10'61
	28	11	22	790'473	45	10'18		6	10	39	012'445	45	10'58
May	29	12	15	791'510	45	10'36		12	10	48	018'451	45	10'33
	3	13	54	795'580	45	10'40		19	7	31	025'313	45	10'20
	6	11	18	798'471	12	10'42		20	8	17	026'345	45	10'07
	9	11	39	801'486	45	10'44		21	9	9	027'381	45	10'00

TABLE II.—*continued.*

Date G.M.T.				Julian Day.	Exposure.	Phot. Magn.	Date G.M.T.				Julian Day.	Exposure.	Phot. Magn.
1911.	h	m		2419	min.		1911.	h	m		2419	min.	
Jan.	1	7	51	038'328	45	10'02	June	14	12	22	202'515	45	10'42
	2	8	4	039'336	45	10'11	July	2	13	4	220'545	45	10'38
	6	12	50	043'535	32	10'21		5	11	54	223'496	40	10'40
	9	9	8	046'380	45	10'20		10	12	11	228'507	45	10'40
	18	8	7	055'337	45	10'30		12	12	11	230'507	45	10'43
	23	8	26	060'351	45	10'40		14	11	36	232'483	33	10'49
	27	9	23	064'391	45	10'38		23	11	15	241'168	31	10'42
	30	10	8	067'422	45	10'39	Aug.	1	11	13	250'467	45	10'55
	31	13	2	068'543	40	10'40		2	13	24	251'559	40	10'54
Feb.	1	7	6	069'296	45	10'43		8	10	36	257'442	45	10'57
	8	10	20	076'431	45	10'42		14	12	34	263'524	45	10'58
	19	8	43	087'364	45	10'48		16	12	4	263'503	45	10'56
	22	10	25	090'435	45	10'54		24	11	29	273'479	45	10'63
	23	9	36	091'400	45	10'47	Sept.	6	10	14	286'427	45	10'66
	28	9	37	096'401	45	10'45		7	10	23	287'433	45	10'55
Mar.	4	10	53	100'454	45	10'47		14	9	37	294'401	45	10'58
	6	11	57	102'498	45	10'50		20	11	9	300'464	45	10'49
	9	10	21	105'431	40	10'56		23	10	21	303'431	45	10'58
	16	8	42	112'363	41	10'55		28	11	56	308'497	45	10'40
	24	8	34	120'357	45	10'59	Oct.	3	12	4	313'503	45	10'32
	25	11	9	121'464	45	10'57		13	8	14	323'343	20	10'27
Apr.	2	11	36	129'483	28	10'61		23	7	20	333'306	45	10'26
	23	10	28	150'437	45	10'65		24	13	30	334'563	45	10'22
	28	13	22	155'557	45	10'58		26	12	23	336'516	45	10'33
May	2	10	40	159'445	45	10'64		30	13	25	340'560	40	10'36
	4	12	55	161'538	45	10'64	Nov.	3	12	31	344'521	35	10'38
	9	11	10	166'465	45	10'60		8	6	39	349'278	45	10'40
	18	11	11	175'465	45	10'48		9	8	16	350'344	45	10'42
	19	11	11	176'465	45	10'49		16	9	0	357'375	40	10'44
	21	12	0	178'500	45	10'42		21	10	29	362'438	45	10'43
	26	11	28	183'478	43	10'38		24	9	13	365'384	45	10'45
	28	11	30	185'479	45	10'40		25	10	10	366'424	45	10'40
	30	11	34	187'482	45	10'37		28	14	58	369'624	45	10'46
	1	11	40	189'486	45	10'35	Dec.	1	12	52	372'536	45	10'43
June	3	11	57	191'498	45	10'35		11	7	13	382'301	45	10'27
	7	11	33	195'481	45	10'32		17	7	26	388'310	45	10'17
	8	11	41	196'487	45	10'32		20	11	54	391'496	45	10'12
	12	11	59	200'499	45	10'35		21	16	46	392'699	40	10'18

TABLE II.—*continued.*

Date G.M.T.	Julian Day.	Exposure.	Phot. Magn.	Date G.M.T.	Julian Day.	Exposure.	Phot. Magn.
1912. h m	2419	min.		1912. h m	2419	min.	
Jan. 7 7 41	409'321	45	10'09	Sept. 12 9 55	658'413	45	9'96
9 11 40	411'486	45	10'15	13 9 27	659'394	45	9'90
14 9 28	416'395	45	10'23	17 8 38	663'360	45	9'83
29 10 19	431'431	45	10'39	21 9 45	667'407	45	9'75
Feb. 9 7 56	442'311	40	10'46	23 8 19	669'347	45	9'70
15 9 19	448'389	45	10'53	Oct. 7 8 28	683'353	45	10'05
20 12 11	453'507	45	10'58	9 7 18	685'305	45	10'07
23 11 45	456'490	45	10'58	14 9 21	690'389	45	10'09
Mar. 5 9 27	467'394	45	10'64	17 9 3	693'377	45	10'33
6 11 11	468'465	45	10'59	18 9 18	694'388	45	10'30
8 10 52	470'453	45	10'57	31 7 30	707'313	45	10'36
13 9 52	475'411	45	10'46	Nov. 1 7 30	708'313	45	10'33
20 8 57	482'373	33	10'34	11 8 1	718'334	45	10'29
22 10 3	484'419	22	10'20	13 7 53	720'329	45	10'25
Apr. 11 9 50	504'410	45	9'98	30 9 28	737'395	45	9'84
16 11 41	509'487	45	9'99	Dec. 2 9 14	739'385	45	9'88
19 9 44	512'407	34	9'93	6 10 28	743'347	40	9'80
23 12 0	516'500	45	9'92	9 14 32	746'605	32	9'75
25 11 5	518'462	24	9'98	11 10 57	748'457	36	9'84
May 7 11 12	530'466	45	10'14	16 9 22	753'390	45	9'96
9 10 33	532'440	45	10'15	31 6 22	768'265	40	10'35
16 11 29	539'479	45	10'23	1913.			
24 11 34	547'482	45	10'28	Jan. 3 9 39	771'403	30	10'41
June 7 11 19	561'472	25	10'47	5 7 17	773'304	45	10'43
11 11 42	565'488	45	10'29	12 7 15	780'302	45	10'50
14 12 11	568'507	45	10'17	15 8 38	783'360	45	10'56
July 3 11 13	587'467	25	10'00	29 7 23	797'308	45	10'48
4 11 24	588'475	45	10'10	Feb. 1 7 5	800'296	45	10'39
13 12 47	597'533	45	10'18	5 8 11	804'340	32	10'35
16 11 10	600'465	45	10'37	10 10 41	809'446	45	10'18
25 10 57	609'457	45	10'43	14 8 19	813'347	45	10'09
Aug. 1 11 23	616'474	45	10'26	26 9 45	825'407	45	9'94
2 10 41	617'446	45	10'25	Mar. 7 7 46	834'324	36	10'08
5 10 47	620'450	45	10'25	11 8 20	838'347	45	10'18
27 10 44	642'449	45	10'12	25 11 33	852'481	32	10'39
Sept. 5 11 18	651'471	45	10'10	Apr. 7 11 8	865'463	45	10'48
6 11 16	652'469	45	10'06	23 9 47	881'408	45	10'69
11 10 18	657'430	45	9'90	28 10 29	886'438	45	10'58

TABLE II —continued.

Date G.M.T.	Julian Day.	Exposure.	Phot. Magn.	Date G.M.T.	Julian Day.	Exposure.	Phot. Magn.
1913 h m	2419	min.		1913 h m	2420	min.	
May 9 10 22	897'432	45	10'34	Aug. 27 10 58	007'458	45	9'85
26 10 41	914'446	40	10'20	Sept. 9 9 35	020'400	40	10'28
June 2 11 14	921'468	45	10'27	29 9 35	040'400	34	10'46
3 11 47	922'491	45	10'20	Oct. 9 10 34	050'441	40	10'52
24 12 57	943'540	45	10'48	21 10 10	062'424	40	10'52
30 11 37	949'484	45	10'64	22 10 44	063'449	40	10'45
July 6 11 14	955'468	40	10'58	29 8 48	070'367	40	10'40
22 10 39	971'445	40	10'40	Nov. 18 6 18	090'263	45	10'07
25 10 53	974'454	33	10'26	Dec. 23 7 9	125'298	40	10'32
29 11 50	978'493	45	10'25	31 11 9	133'464	40	10'47
Aug. 1 11 11	981'465	45	10'02				
11 12 26	991'518	45	9'80	1914			
	2420			Jan. 26 7 51	159'328	45	10'15
25 11 7	005'462	45	9'90	Apr. 15 11 14	238'468	45	10'27

4. It is believed that no attempt to represent the variations of a star of this type has been even remotely successful. The present observations have therefore received careful discussion. As a starting-point, a 20-day ephemeris was formed. With the object of smoothing out some of the more rapid fluctuations, the M process* was applied to the table. This consists simply in writing down the means of the preceding and following entries. The process was repeated eight times in succession, and resulted in a remarkably smooth and simple curve, suggesting that only terms of long period had survived the operation. The reason for this will be seen more clearly later. At this stage the reductions had not been completed, and from the material available it appeared that the slow variation could be represented by a single term of about 960 days' period. Accordingly, the observations were roughly analysed for this period in the search for dependent variations. The more significant terms thus obtained were these:—

Period	960	160	107	96	87	80 days
Amplitude	0'15	0'12	0'09	0'12	0'08	0'08 mag.

Three successive periods of 480 days were next treated in the same way and gave the following terms:—

Period	160	120	96	80	60	40 days
Amplitude	0'13	0'13	0'10	0'11	0'06	0'08 mag.

These preliminary results were obtained very quickly and easily

* Cf. *M.N.*, lxxvi. p. 60.

and they contain a considerable amount of information which has been confirmed by closer study. It may be summarised thus:—

(i.) There is at least one term of long period, of the order of 960 days.

(ii.) There is a term of about 160 days.

(iii.) There is a group of terms with periods from 80 to 100 days.

(iv.) There may be terms of shorter period, as 60 and 40 days.

5. The most important point suggested is (iii.), the group of periods lying between 80 and 100 days. Such a group of neighbouring terms presents a notorious difficulty. It was finally decided to make the investigation in a systematic way by calculating the periodogram between the limits 80 and 160 days. For this purpose a 4-day ephemeris was prepared, covering the time over which the observations extended. The initial epoch adopted throughout is

$$T = 1909 \text{ Nov. } 7 = \text{J.D. } 2418618.$$

Every period at intervals of 4 days from 80 to 160 days was analysed, and as the work was not greatly increased by calculating the second harmonic, every period at intervals of 2 days from 40 to 80 days was examined at the same time. With a little extension suggested by the outcome, the results are given in Table III. This shows the period of analysis (P), the number (n) of periods treated, the amplitude (r) of the corresponding variation, and the phase (α) of each term referred to the above epoch T . The form of the typical term is here taken to be

$$r \sin\{2\pi(t - T)/P + \alpha\}.$$

Periods which give rise to amplitudes of at least a tenth of a magnitude are marked by special type. It is thus evident that the chief features of the periodogram are a band between 80 and 96 days and another in the neighbourhood of 160 days. The periodicity near 120 days suggested by the analysis of three periods of 480 days is not conspicuous, and may be classed with the minor variations now located at 46 and 60 days.

TABLE III.

Period.	No.	Amplitude.	Phase.	Period.	No.	Amplitude.	Phase.
d		m		d		m	°
40	36	·02	(180)	56	26	·01	...
42	36	·02	41	58	26	·02	94
44	35	·01	...	60	24	·04	302
46	32	·04	299	62	24	·01	223
48	32	·01	...	64	24	·01	346
50	30	·02	151	66	22	·02	324
52	28	·00	...	68	22	·01	152
54	28	·02	21	70	22	·03	266

TABLE III.—*continued.*

Period. d	No.	Amplitude. m	Phase.	Period. d	No.	Amplitude. m	Phase.
72	20	·02	316°	116	13	·05	333°
74	20	·00	...	120	12	·05	85
76	20	·04	310	124	12	·03	179
78	18	·04	83	128	12	·01	99
80	18	·11	252	132	11	·03	152
82	18	·17	343	136	11	·03	235
84	18	·11	70	140	11	·00	...
86	16	·06	281	144	10	·03	213
88	16	·16	357	148	10	·07	253
90	16	·14	56	152	10	·11	305
92	16	·07	79	156	9	·12	343
94	16	·09	62	160	9	·12	26
96	16	·12	109	164	9	·10	63
98	15	·08	162	168	9	·06	80
100	15	·02	263	172	8	·06	97
104	14	·04	103	176	8	·03	80
108	14	·03	128	180	8	·04	83
112	13	·04	199	188	8	·03	130

6. Table III. contains all the information available with regard to the shorter periods, though the question of interpretation is another matter. Deferring this, we return to the consideration of the long periods. If the variation of magnitude may be represented by

$$m = \Sigma r \sin (\mu t + a),$$

and the magnitudes are tabulated with the common interval τ , we deduce by the M process,

$$M. m = \Sigma r \cos \mu \tau \sin (\mu t + a)$$

$$M^s. m = \Sigma r \cos^s \mu \tau \sin (\mu t + a).$$

In order to deduce the slower changes independently of the more rapid terms, it is above all necessary to eliminate the effect of the periods in the neighbourhood of 80 days. To calculate these terms directly is not altogether easy or safe. But if we choose 20 days for τ , as mentioned in § 4, $\mu \tau$ is near $\frac{1}{2}\pi$ and $\cos \mu \tau$ is very small for any term of this class. Similarly, for any term with period near 160 days, $\mu \tau$ is near $\frac{1}{2}\pi$, and thus the amplitude will be diminished considerably, though not so rapidly as terms of 80 days. This explains why the repetition of the M process leads to the elimination of the short periods and leaves the longer periods, the effect of which can be studied separately. A graphical representation of $M^s. m$ is shown in fig. 2.

As a matter of detail it is to be noticed that the process treats the systems of alternate numbers quite independently. Also the very short periods (as 40 days) may persist. For both reasons a slight irregularity was found between the alternate systems, which in the present case was finally smoothed out by twice taking the means of consecutive entries in the table. When 10 days is adopted as the interval of the tabular argument, the local irregularities due to the very short periods are not noticeable. But then, as might be anticipated, the periodicity of 160 days is not eliminated. Indeed, this particular operation yields the simplest and clearest proof of a periodicity near 155 days.

7. Fig. 2 shows unmistakably that the slower variation can-

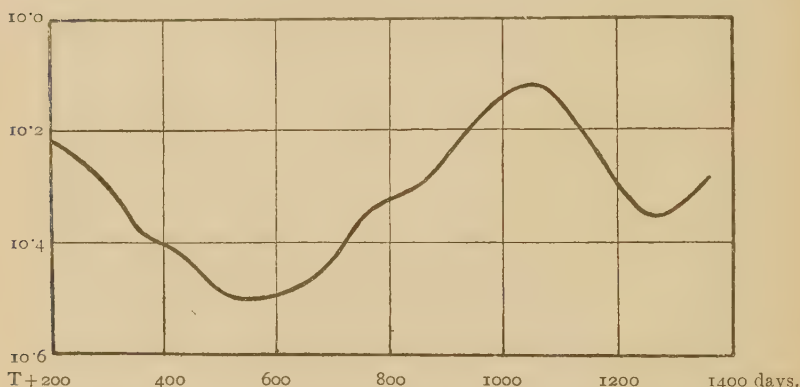


FIG. 2.

not be represented by a single term of long period. An attempt was therefore made to represent the graph by two terms. Dale's method was used as particularly appropriate in the circumstances, a tabular interval of 200 days being adopted. One root of the quadratic gave $\cos \mu\tau = +.265$ or $\mu\tau = 74.6^\circ$. With the unit employed this means a period of 965 days, evidently the period already suggested in § 4. But the other root gave $\cos \mu\tau = -2.2$, which cannot correspond with a real period. Though a representation by a series of empirical harmonic terms might doubtless be forced with a little trouble and ingenuity, it seems better to admit frankly that the present material is inadequate for a satisfactory discussion of the terms of long period. It must suffice to have established their existence and to have shown something of the nature of their effects.

It seems possible that the imaginary period thus presented may really point to something of the nature of a dissipative function in the energy of the star. In this connexion allusion may be made to a paper* by Sir J. J. Thomson on "A method of finding the coefficients of absorption of the different constituents of a beam of heterogeneous Röntgen rays, or the periods and coefficients of

* *Phil. Mag.*, xxx, p. 780 (1915).

damping of a vibrating dynamical system," which, though dealing primarily with very different subject-matter, presents a curious parallel in method and in time with the note* by one of us on Professor Dale's method of finding periodicities.

A feature which will not escape notice in the light variations of RS Camelopardalis is that during the four and a half years covered by the present observations the limits of the range of one magnitude (from 9.7 to 10.7) were practically never overstepped. In itself this fact suggests remarkable steadiness, and may be opposed to any idea of appreciable secular change or considerable unsuspected fluctuations of long period. But the interval is not long, and as a matter of fact there is ample reason for caution. In the notice announcing discovery it is remarked that "Son éclat varie de 8.4 à 9.2 gr. à en juger d'après les 26 clichés obtenus en 1897-1905." The magnitude scale here may be different from ours. But on the face of it this is a suggestion that the mean magnitude is changing, and indeed that the star has grown fainter in very recent times.

For the purpose of prediction the terms of long period must be definitely ascertained. Otherwise, any success in this direction must be a matter of luck rather than of sound deduction. Hence a long and continuous series of observations is absolutely necessary for further progress. Extreme accuracy in the individual magnitudes is of less importance than continuity over a long interval. An observer who is prepared to make a careful determination of the magnitude of RS Camelopardalis at least once a week for ten or twenty years without a break should find this star well worthy of study.

8. We now return to the periodicities indicated in Table III. The chief task is to interpret the groups in the neighbourhood of 80 and 160 days. If a single harmonic variation represented by $H \sin(q\theta + \beta)$ is analysed for the period 2π , so that

$$q = \frac{\text{period of analysis}}{\text{period of term analysed}}$$

and n periods 2π enter into the analysis, the result is a term $K \sin(\theta + \alpha)$, where

$$K \sin \alpha = \frac{2q \sin n(q-1)\pi}{n\pi(q^2-1)} \cdot H \sin\{n(q-1)\pi + \beta\},$$

$$K \cos \alpha = \frac{2 \sin n(q-1)\pi}{n\pi(q^2-1)} \cdot H \cos\{n(q-1)\pi + \beta\}.$$

In the case which is of the greatest importance in the interpretation of the periodogram q is nearly 1, and then approximately

$$K = H \cdot \frac{\sin n(q-1)\pi}{n(q-1)\pi} \quad . \quad . \quad . \quad (1)$$

$$\alpha = \beta + n(q-1)\pi \quad . \quad . \quad . \quad (2)$$

* *M.N.*, lxxvi. p. 60.

These expressions give the "diffraction" effects in the neighbourhood of a pure periodicity. The amplitude is used throughout, not its square, which is the analogue of the optical intensity.

9. The interpretation of the band at 160 days seems simple and definite. A pure oscillation of amplitude $0^m.12$ in the period 158 days is assumed. Its effects when analysed for neighbouring periods are calculated in Table IV. Here K is calculated by (1),

TABLE IV.

P.	$q-1$.	n .	$n(q-1)\pi$.	K .	$^\circ K_0$.	K_0-K .	a .	a_0 .	a_0-a .
d									
144	- '0886	10	- 159	'02	'03	+ '01	205	213	+ 8
148	- '0633	10	- 114	'05	'07	+ '02	250	253	+ 3
152	- '0380	10	- 68	'10	'11	+ '01	296	305	+ 9
156	- '0127	9	- 21	'12	'12	'00	343	343	0
160	+ '0127	9	+ 21	'12	'12	'00	25	26	+ 1
164	+ '0380	9	+ 62	'10	'10	'00	66	63	- 3
168	+ '0633	9	+ 103	'06	'06	'00	107	80	- 27
172	+ '0886	8	+ 128	'04	'06	+ '02	132	97	- 35

in which $H = 0.12$, and a by (2), in which β is taken to be $+4^\circ$. The quantities n , K_0 , and a_0 are taken from Table III. The comparison of the amplitudes— K , K_0 —and the phases— a , a_0 —shows that any residual effects not explained by the simple oscillation are merely of the order of the terms to be met with anywhere in the quiescent parts of the periodogram. In fact, this seems an excellent illustration of the theory of the periodogram in relation to "homogeneous radiations."

Hence it is possible to adopt as one term in the variation of the star

$$0.12 \sin\{2\pi(t-T)/158+4^\circ\} \quad . \quad . \quad . \quad (3)$$

The constants of this term are naturally susceptible of improvement, but of its definite character there seems little doubt.

10. The meaning of the band near 80 days is clearly much more complicated and difficult. But a term at 82 days is plainly indicated. In Table V. this term is treated in the same way as the term of 158 days in Table IV.

TABLE V.

P.	$q-1$.	n .	$n(q-1)\pi$.	K .	K_0 .	K_0-K .	a .	a_0 .	a_0-a .
d									
74	- '0976	20	- 351	- '00	'00	'00	352	...	...
76	- '0732	20	- 264	- '04	'04	...	79	310	...
78	- '0488	18	- 158	+ '02	'04	...	185	83	...
80	- '0244	18	- 79	+ '12	'11	- '01	264	252	- 12
82	0	18	0	+ '17	'17	'00	343	343	0
84	+ '0244	18	+ 79	+ '12	'11	- '01	62	70	+ 8
86	+ '0488	16	+ 141	+ '04	'06	...	124	281	...
88	+ '0732	16	+ 211	- '02	'16	...	194	357	...
90	+ '0976	16	+ 281	- '03	'14	...	264	56	...

The sign of K can be made positive by adding π to a . Hence the table shows that the residual terms for 76 and 78 days are not large, that the variations found for the periods 80, 82, and 84 days are well explained simultaneously by a single term of the intermediate period, and that the terms of longer period in the neighbourhood are slightly modified when the single term is removed. This term is taken to be

$$0.17 \sin\{2\pi(t-T)/82 + 343^\circ\} \quad . \quad . \quad . \quad (4)$$

11. A similar treatment may be given to the term of 96 days on the other side of the band. This is done in Table VI.

TABLE VI.

P.	$(q-1)^{-1}$.	n .	$n(q-1)\pi$.	K .	K_0 .	$K_0 - K$.	a .	a_0 .	$a_0 - a$.
d									
88	-12	16	-240	-.02	.16	...	229	357	...
90	-16	16	-180	.00	.14	...	289	56	...
92	-24	16	-120	+.05	.07	...	349	79	...
94	-48	16	-60	+.10	.09	-.01	49	62	+13
96	...	16	0	+.12	.12	.00	109	109	0
98	+48	15	+56	+.10	.08	-.02	165	162	-3
100	+24	15	+112	+.06	.02	-.04	221	263	+42

The terms found for the periods 94, 96, and 98 days are here fairly explained by a single term

$$0.12 \sin\{2\pi(t-T)/96 + 109^\circ\} \quad . \quad . \quad . \quad (5)$$

The terms of shorter period are modified, and a small residual term of 100 days is left. Table III. shows that the latter belongs to a band of slight intensity which extends as far as 120 days, and could apparently be resolved into a pair of terms, each with an amplitude of the order of .05 and with periods 102 and 118 days. But these are too small to receive closer examination.

12. The modifications in the periodogram which follow from the removal of the terms (4) and (5) are found by subtracting from $K_0 \sin a_0$, $K_0 \cos a_0$ the values of $K \sin a$, $K \cos a$ derived from Tables V. and VI. The result is to give new values K' , a' which are entered in Table VII. This has the same form as the preceding, and shows the effect of a third term with the period 89 days.

TABLE VII.

P.	$q-1$.	n .	$n(q-1)\pi$.	K .	K' .	$K' - K$.	a .	a' .	$a' - a$.
d									
82	-.0787	18	-255	-.03	.00	+.03	128	...	...
84	-.0562	18	-182	.00	.02	+.02	201	207	+6
86	-.0337	16	-97	+.09	.09	.00	286	288	+2
88	-.0112	16	-32	+.14	.17	+.03	351	357	+6
90	+.0112	16	+32	+.14	.12	-.02	55	48	-7
92	+.0337	16	+97	+.09	.09	.00	120	117	-3
94	+.0562	16	+162	+.02	.03	+.01	185	180	-5
96	+.0787	16	+227	-.03	.00	+.03	250	...	...

The amplitude of the term adopted is 0.15 and the phase β is 23° , *i.e.* the term is

$$0.15 \sin\{2\pi(t - T)/89 + 23^\circ\} \quad . \quad . \quad . \quad (6)$$

Thus the effect of admitting the three terms with periods 82, 89, and 96 days is to leave outstanding in the periodogram no ordinate greater than .05, while at the same time the phases corresponding to the principal ordinates are all well represented. The effective terms coincide practically with the maximum ordinates of the periodogram, and the further close examination may have seemed superfluous. But without it the interpretation of the periodogram in this region would have been quite insecure. The structure of this periodogram seems to be comparatively simple. But it is not pretended that the analysis of it given here is either exact or complete. It should be enough if the main features of the variation have been extracted, and it is hoped that this has been done successfully.

13. It has been pointed out that the series of observations is not long enough to determine the terms of long period, and that prediction of the course of future variations is therefore not feasible. Partly for this reason minor terms with amplitudes of an order not exceeding .05 have been disregarded. The object of surveying the problem presented by the star in its wider aspects has been kept strictly in view, rather than of constructing an elaborate formula which will represent the interpolated curve with minute accuracy, but would be unworthy of confidence outside the dates covered by the observations. Nevertheless a direct comparison of the terms which have been found with the observed curve may be of interest, and is accordingly given in Table VIII. The calculated magnitudes are obtained by applying the four terms (3), (4), (5), and (6) to values which were read from fig. 2 at intervals of 10 days. The number of days given in the first column is the time reckoned from the epoch T given in § 5. Owing to its method of derivation, fig. 2 does not extend to the beginning or the end of the observations, and Table VIII. is similarly incomplete. The observed magnitudes are read from a curve similar to fig. 1. This curve is by no means faultless, and may be responsible for some of the discrepancies. But for the most part the residuals are probably attributable to the accumulation of those smaller periodicities which have been deliberately left aside. The numerical average of the residuals is $^m.087$.

TABLE VIII.

T+.	m(C).	m(O).	O - C.	T+.	m(C).	m(O).	O - C.
d				d			
200	10.67	10.64	- .03	590	10.41	10.32	- .09
210	.40	.42	+ .02	600	.37	.35	- .02
220	10.07	.17	+ .10	610	.34	.39	+ .05
230	9.87	10.03	+ .16	620	.33	.45	+ .12
240	9.84	9.98	+ .14	630	.37	.50	+ .13
250	10.01	10.07	+ .06	640	.47	.56	+ .09
260	.28	.18	- .10	650	.59	.62	+ .03
270	.48	.33	- .15	660	.70	.67	- .03
280	.55	.44	- .11	670	.74	.64	- .10
290	.45	.38	- .07	680	.67	.53	- .14
300	.23	.20	- .03	690	.53	.40	- .13
310	10.05	.07	+ .02	700	.35	.27	- .08
320	9.99	.08	+ .09	710	.24	.26	+ .02
330	10.11	.25	+ .14	720	.20	.33	+ .13
340	.35	.43	+ .08	730	.26	.40	+ .14
350	.62	.55	- .07	740	.35	.43	+ .08
360	.78	.62	- .16	750	.43	.45	+ .02
370	.78	.66	- .12	760	.44	.37	- .07
380	.62	.67	+ .05	770	.37	.19	- .18
390	.40	.62	+ .22	780	.26	.07	- .19
400	.20	.37	+ .17	790	.17	.08	- .09
410	.11	.01	- .10	800	.17	.20	+ .03
420	.14	.07	- .07	810	.27	.32	+ .05
430	.25	.23	- .02	820	.42	.43	+ .01
440	.39	.33	- .06	830	.57	.53	- .04
450	.49	.41	- .08	840	.61	.63	+ .02
460	.55	.45	- .10	850	.53	.62	+ .09
470	.51	.48	- .03	860	.36	.42	+ .06
480	.49	.52	+ .03	870	.16	.21	+ .05
490	.49	.54	+ .05	880	.04	10.06	+ .02
500	.48	.57	+ .09	890	.02	9.97	- .05
510	.54	.60	+ .06	900	.09	9.98	- .11
520	.57	.63	+ .06	910	.20	10.09	- .11
530	.59	.66	+ .07	920	.27	.23	- .04
540	.58	.66	+ .08	930	.27	.37	+ .10
550	.54	.58	+ .04	940	.24	.45	+ .21
560	.51	.46	- .05	950	.12	10.18	+ .06
570	.47	.37	- .10	960	.09	9.96	- .13
580	10.44	10.32	- .12	970	10.13	10.05	- .08

TABLE VIII.—continued.

T+.	m(C).	m(O).	O-C.	T+.	m(C).	m(O).	O-C.
d				d			
980	10'24	10'28	+ '04	1190	10'25	10'22	- '03
990	'36	'44	+ '08	1200	9'98	9'99	+ '01
1000	'40	'27	- '13	1210	'36	9'98	+ '12
1010	'34	'14	- '20	1220	9'91	10'16	+ '25
1020	'20	'10	- '10	1230	10'14	'31	+ '17
1030	10'00	10'10	+ '10	1240	'44	'43	- '01
1040	9'87	9'91	+ '04	1250	'65	'55	- '10
1050	'86	'74	- '12	1260	'68	'65	- '03
1060	9'96	9'87	- '09	1270	'55	'57	+ '02
1070	10'13	10'13	'00	1280	'32	'35	+ '03
1080	'25	'34	+ '09	1290	'11	'22	+ '11
1090	'28	'36	+ '08	1300	'07	'21	+ '14
1100	'20	'30	+ '10	1310	'19	'30	+ '11
1110	10'07	10'15	+ '08	1320	'41	'43	+ '02
1120	9'99	9'87	- '12	1330	'63	'58	- '05
1130	10'02	9'76	- '26	1340	'69	'62	- '07
1140	'19	10'09	- '10	1350	'59	'47	- '12
1150	'43	'34	- '09	1360	'30	10'18	- '12
1160	'60	'52	- '08	1370	10'01	9'82	- '19
1170	'65	'57	- '08	1380	9'80	'76	- '04
1180	10'48	10'48	'00	1390	9'78	9'94	+ '16

14. The group of three terms (4), (5), (6), may be written
 $0'17 \sin (4^{\circ}39' t' - 17^{\circ}) + 0'15 \sin (4^{\circ}045' t' + 23^{\circ})$
 $+ 0'12 \sin (3^{\circ}75' t' + 109^{\circ})$

where $t' = t - T$. With a little adjustment these may be written

$$0'172 \sin (4^{\circ}382' t' - 25^{\circ}) + 0'147 \sin (4^{\circ}062' t' + 38^{\circ})$$

$$+ 0'122 \sin (3^{\circ}742' t' + 101^{\circ}),$$

and it will be noticed that this expression is of the form

$$2a \sin 2\theta + (a+b) \sin (\theta + \phi) + 2b \sin 2\phi$$

$$= (a+b) \sin (\theta + \phi) \{1 + 2 \cos (\theta - \phi)\}$$

$$+ 2(a-b) \cos (\theta + \phi) \sin (\theta - \phi).$$

Hence the group of adjusted terms becomes

$$0'15 \sin (4^{\circ}062' t' + 38^{\circ}) \{1 + 2 \cos (0^{\circ}32' t' - 63^{\circ})\}$$

$$+ 0'05 \cos (4^{\circ}062' t' + 38^{\circ}) \sin (0^{\circ}32' t' - 63^{\circ}).$$

Now this represents a simple oscillation in the period 88.7 days, with slowly changing amplitude and phase. The square of the amplitude is

$$\{0'15 + 0'30 \cos (0^{\circ}32' t' - 63^{\circ})\}^2 + \{0'05 \sin (0^{\circ}32' t' - 63^{\circ})\}^2,$$

and the change of phase ψ is given by

$$\tan \psi = \frac{\sin (0^{\circ}.32 t' - 63^{\circ})}{3 + 6 \cos (0^{\circ}.32 t' - 63^{\circ})}.$$

Both amplitude and phase recur after about 1150 days, in which time about 13 of these varying oscillations have taken place.

Very likely the periods with which we are actually dealing are not commensurable. The importance of this ideal representation of the most interesting feature in the variation of a particular star is the light which it throws on the nature of long-period variation in general. It is a common thing to find that the maxima of long-period variables do not recur at equal intervals, and probably such maxima are also unequal in intensity. The familiar device which serves the purposes of prediction more or less satisfactorily is to add one or more sine terms to the date. The reason is to be found partly in practical convenience and partly in the altogether undue emphasis which has been laid on this phase of the light curve. But if a long and complete series of observations were available for thorough discussion, it is probable that the whole of the facts would be represented by the superposition of a number of terms, each with an unvarying period.

It is unsafe to generalise from a particular instance which at first sight looked extremely puzzling, but may be comparatively simple. The variation of RS Camelopardalis presents no evidence of discontinuity.

Dunsink Observatory:

1916 April 12.

Note on General Precession. By H. C. Plummer, M.A.

This note is concerned with a simple matter of definition. General precession is a term of such fundamental importance that a clear and simple description should not be difficult to find. But reference to a number of admirable text-books suggests that a student may have quite considerable difficulty. The definitions given are not merely vague, but they are too often accompanied by remarks which are actually misleading. In some cases this would not matter very much, because a quantity is often defined implicitly with sufficient precision by the manner in which it is derived, and by the result obtained. This does not apply to general precession, because this is generally expressed by a series which stops at the term involving the second power of the time. Thus the definition may be varied considerably without altering the result to be deduced.

Presumably the following may be suggested as a standard definition. General precession at the time t is the (negative)

mean longitude for the epoch 1850.0 (or whatever the fundamental epoch may be) of the mean equinox at the epoch t . This is certainly definite, and allows the position of the mean equinox to be calculated with all the accuracy with which the positions of the mean equator and ecliptic are known at any time.

Since the mean equinox is a point not far removed from the fixed ecliptic, general precession may be very roughly described as the displacement of the mean equinox along the fixed ecliptic. But this must be properly understood, and is in no sense a definition.

Let the position of the moving ecliptic be given by

$$\begin{aligned}i \sin \Omega &= gt + ht^2 \\ i \cos \Omega &= g't + h't^2\end{aligned}$$

and the luni-solar precession in longitude by

$$\psi = \alpha t + \beta t^2.$$

Then the general precession is

$$\psi' = Pt + P't^2$$

where

$$\begin{aligned}P &= \alpha - g \cot \epsilon_0 \\ P' &= \beta - (ag' + h) \cot \epsilon_0 + gg' \cot^2 \epsilon_0\end{aligned}$$

and ϵ_0 is the mean obliquity of the ecliptic. This expression for P' is, of course, familiar enough.

But unfortunately, as it seems, another idea has become associated with the general precession. This refers to the displacement of the mean equinox along the moving ecliptic. It is at once evident that this is meaningless unless the origin of the displacement is accurately defined. Here a natural difficulty arises, and hence a number of loose statements which are apt to be misleading to the reader.

Bessel (*Tab. Reg.*, p. iv) refers to “punctorum motum, in quibus Aequator a vera Ecliptica scindatur, i.e. *Præcessionem generalem*.” This is not very helpful. Brünnow (*Spherical Astronomy*, 1865, p. 117) uses the words, “If, further, D' denotes the same point of the true ecliptic which in the fixed ecliptic was denoted by D (the equinox of the fundamental epoch), the arc CD' is the general precession.” But it is quite impossible to see how “the same point” on a moving great circle is to be identified. Ball (*Spherical Astronomy*, p. 176) reproduces Brünnow’s figure, but draws DD' perpendicular to CD' (the moving ecliptic). This is definite, but inaccurate.

Tisserand (*Méc. Cé.*, tome II., p. 437) in a “Remarque” states the point with perfect clearness. He then defines the point on the moving ecliptic from which general precession is to be reckoned as the point which is equidistant with the mean equinox on the fixed ecliptic from the common node of the two ecliptics. This implies that the fundamental system of ecliptic axes can be brought into coincidence with the moving system by a simple rotation about the line of nodes. Accordingly the

general precession can be deduced from the luni-solar precession by a reverse rotation of the N. pole of the mean equator about the same line. But the result, which is easily found to follow from this assumption, gives

$$P' = \beta - (ag' + h) \cot \epsilon_0 + (\tfrac{1}{2} + \cot^2 \epsilon_0) gg'$$

which involves the extraneous term $\frac{1}{2} gg'$. The discrepancy presented by this term is the cause of the present note. Tisserand's statement is qualified by the remark "L'angle i étant très petit, on a, à fort peu près, . . ." But such statements might be expected to be true within the limits contemplated, *i.e.* to the second order in i (or t). It is true that the discrepancy is of trifling practical importance, but the reason for this is not merely that i is small (for this applies equally to an associated term), but that Ω is near 180° .

It is easy to specify the point on the moving ecliptic from which the general precession ought to be reckoned if accuracy to the second order is considered sufficient. For this is simply the intersection of the moving ecliptic with the great circle passing through the fixed equinox and the pole of the fixed ecliptic.* Let Ω' be the distance of this point from the node. Then

$$\tan \Omega' = \tan \Omega \sec i$$

so that

$$\begin{aligned} \tan (\Omega' - \Omega) &= \tan \Omega (\sec i - 1) / (1 + \tan^2 \Omega \sec i) \\ &= \tfrac{1}{2} i^2 \sin \Omega \cos \Omega = \tfrac{1}{2} gg' t^2 \end{aligned}$$

to the order named. Hence the motion measured from this point along the moving ecliptic will give the required general precession of the mean equinox, but it is not the equivalent of the definition suggested above.

Bauschinger (*Bahnbestimmung*, p. 69), after finding the ordinary expressions, adds a statement which may be expressed in the form that because i is small, the general precession is equal to the motion of the equinox along the moving ecliptic, reckoned from the projection on this ecliptic of the fixed equinox. This is the equivalent of the statement given by Ball as a definition. Let Ω'' be the projection of the arc Ω on the moving ecliptic, so that

$$\tan \Omega'' = \tan \Omega \cos i,$$

then

$$\begin{aligned} \tan (\Omega' - \Omega'') &= \tan \Omega (\sec i - \cos i) / (1 + \tan^2 \Omega) \\ &= i^2 \sin \Omega \cos \Omega = gg' t^2 \end{aligned}$$

to the second order. Hence $\Omega'' - \Omega' = 2 (\Omega - \Omega')$, and the error is twice as great as when the displacement is measured relatively to the node. All this of course is geometrically obvious.

* I had written the fixed equinoctial colure. But what is the equinoctial colure? According to Brünnow and L. de Ball this circle belongs to the ecliptic system. But according to Lalande, Delambre, Chauvenet, and Ball it belongs to the equatorial system. Which is right?

year 1915 are summarised, for the first time, on the basis of the recommendations of the International Solar Union. In accordance with these suggestions, only those prominences which reached the height of 30" or greater have been included, and the positions are indicated by angles measured from the north pole of the sun through east, south, and west.

Only 58 observations were made during the year, being fewer than usual because of so many cloudy days in the earlier and later months of the year.

The number of prominences reaching the height considered most appropriate for statistical purposes by the Solar Union was 262: they were classified as "Bright" and "Faint" at the time of observation, and the effect of the new limit of height is to eliminate a large number of both classes from the record.

In the diagram, each bright prominence is represented by the letter B at the corresponding position angle, and each faint prominence by a dot. The distribution in heliographic latitude is shown very clearly in this way. It will be seen that the southern hemisphere exhibited the greatest activity, and that the prominences on the western limb were somewhat more numerous than those on the eastern.

Black absorption markings were noticed seventeen times on bright, and three times on faint, prominences.

The instrument employed was a grating spectroscope, used in conjunction with a 9-in. Cooke refractor.

The Efficiency of Sun-spots in relation to the Mean Daily Range of Terrestrial Magnetic Declination. By Rev. A. L. Cortie, S.J.

Investigations with regard to the efficiency of sun-spots in terrestrial magnetic phenomena have hitherto been mainly concerned with the correlation of sun-spot areas with magnetic phenomena. In former papers of this series I have shown that variation in magnetic phenomena, and particularly in magnetic disturbances, is a function not only of area but also of the position of sun-spots relatively to the sun's equator. In particular, in my last paper on this subject (*M. N.*, lxxvi, 15) additional evidence was adduced in support of the view that magnetic disturbances depend largely upon the heliographic latitudes of the sun-spots. This efficiency is, in general, found to diminish with increasing distance from the solar equator, though a particularly active spot, at a considerable distance from the equator, may produce a disturbance, on account of its influence not being restricted to any one direction.

In the present paper the values of the mean daily range of magnetic declination, or horizontal direction, the mean daily

disc-area, or projected areas of the sun-spots, and the mean heliographic latitudes of the spots for the northern and southern hemispheres are compared for the years 1900-1912, which embrace a solar cycle. The mean daily range of the declination for the period is derived from the yearly tables published by Father Sidgreaves in the Stonyhurst annual reports. The mean daily range for any year is the difference between the means of the highest and lowest daily readings for that year. The disc-areas of the spots are also derived from the Stonyhurst yearly reports, and the means for the successive years are expressed in terms of the $\frac{1}{5000}$ of the visible surface. The mean yearly heliographic latitudes for the spots north and south of the sun's equator respectively have been taken from the Greenwich photo-heliographic results. In these results the heliographic latitudes of the spots for each day are multiplied by their areas, and the sums of the products thus formed are divided by the sums of the corresponding areas to form the mean heliographic latitudes north and south of the equator. These figures therefore are functions not only of the positions of the spots but also of their areas.

The values of the four quantities discussed have been smoothed by twice taking in succession the means of three observed values. They are set forth in the following table, which gives the unsmoothed and the smoothed values, and in the curves plotted from the smoothed results :—

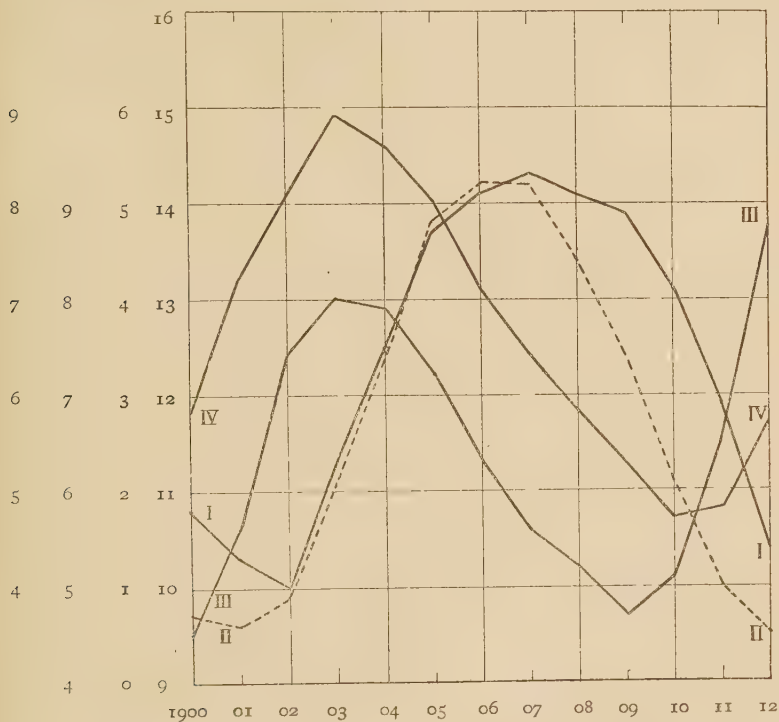
Year.	Mean Daily.				Mean Heliographic Spot Latitudes.			
	Range.	Declin.	Spot Area.		N.		S.	
			$\frac{1}{5000}$		°		°	
1900	9·8	10·8	0·55	0·70	6·61	9·0	8·34	11·6
1901	9·1	10·3	0·29	0·56	8·59	11·2	16·27	14·2
1902	9·0	10·0	0·33	0·92	18·81	14·7	15·29	16·2
1903	11·8	11·3	1·93	2·06	18·11	16·1	21·15	17·8
1904	11·9	12·5	2·54	3·36	16·33	15·7	16·88	17·1
1905	15·0	13·7	6·80	4·76	11·66	14·4	15·55	16·0
1906	13·9	14·1	4·80	5·19	13·98	12·5	14·01	14·2
1907	14·7	14·3	5·80	5·19	10·12	11·1	13·77	12·8
1908	14·1	14·1	4·60	4·39	10·42	10·3	10·34	11·5
1909	13·5	13·9	3·80	3·36	9·45	9·4	9·92	10·6
1910	14·5	13·1	1·80	2·05	8·55	10·2	11·18	9·3
1911	12·7	11·9	0·33	0·99	7·89	12·9	5·99	9·5
1912	8·1	10·4	0·22	0·45	20·53	17·2	7·78	12·2

The smoothed curves drawn from the numbers in the Table, are I. mean daily range in magnetic declination, II. mean daily sun-spot disc-area, III. mean N. heliographic latitude, and IV. mean S. heliographic latitude of the spots. To bring III.

and IV. to approximately the same scale as I. and II., the numbers corresponding to them in the table have been halved. Also IV. has been shifted above III., the base line being changed for the sake of clearness.

With regard to I. and II. (the broken line curve) the spot-area rises in value for 1901-02, while the declination daily range falls. This lag of D after S, or I. after II., is true also of the unsmoothed values of the quantities. From 1902 to 1905 the two curves are practically conterminous, each also reaching a

IV. III. II. I.
Lat. S. Lat. N. Area. D



maximum in 1907. After maximum the fall in S is much more rapid than in D, a further evidence of the lagging of D after S.

The curves III. and IV. for mean heliographic latitude of the spots N and S are very similar, the maximum in both being reached in 1903. The N. latitude curve III. begins to rise again in 1909 very rapidly, a much less rapid rise in that of the S. latitude curve IV. following in 1910. This is evidence of what generally occurs, that a new solar cycle commences with outbursts of spots in high latitudes in the N. hemisphere.

But the noteworthy point is the difference of phase between the latitude curves and those for S and D, the maxima of the former being about four years in advance of the latter. Hence they are falling toward the equator when D is rising to a maximum.

From this it follows that the efficiency of sun-spots in relation to the mean daily range in magnetic declination is greatest when the sun-spots are falling towards the equator. This conclusion is in entire accord with the results established in former papers with regard to sun-spots and magnetic storms. From the curves it would appear that the maximum effect of sun-spots with regard to the daily range of declination is between the mean latitudes 10° - 15° .

Stonyhurst College Observatory:
1916 May 5.

The Surface Currents of Jupiter during the Apparition of 1915-16.
By Scriven Bolton.

The work of determining the longitudes of markings on Jupiter extended over the period 1915 September 9, to 1916 February 9, inclusive. The opposition of Jupiter occurred on 1915 September 17.

In all, 2139 transits of light and dark markings were obtained. This number includes several estimated transits of objects which were not actually observed to cross the central meridian, and such estimates have, as in previous years, proved in many cases of value in the final reduction. Help has been afforded in the matter of identification of the various markings by obtaining a sketch of every object seen to cross the meridian. Incidentally 484 such sketches were accumulated at the close of the apparition.

The observations were made at Bramley, near Leeds, with the 26-in. reflector, which is driven by a powerful clock supplied and fitted by Mr. Calver. The instrument was available for planetary work only for a limited period each night, otherwise the observations might have been more numerous than indicated. In these observations valuable assistance has been rendered by Mr. A. Burnet, and in the annexed table are given his figures of the respective periods of rotation as deduced from the observations:—

Phenomena.	Average date of first observation.	Average date of last observation.	Number of selected spots.	Mean number of observations.	Mean rotation period.
Dark spots and streaks, S.S. Temperate Band	1915 Sept. 26	1916 Jan. 1	14	5	hr. min. sec. 9 55 4'3
White spots, S. Temperate Zone	Oct. 10	Jan. 5	25	7	9 55 18'6
Dark spots, S. Temperate Band	Oct. 6	Jan. 1	23	8	9 55 20'6
White spots, S. Tropical Zone	Oct. 16	Jan. 8	16	7	9 55 36'0
Red Spot Hollow, <i>p</i> and <i>f</i> Shoulders	Sept. 11	Jan. 28	2	20	9 55 36'7
White spot in S. Tropical Disturbance.	Sept. 30	1915 Nov. 20	1	6	9 55 29'6
S. Tropical Disturbance, <i>p</i> end	Sept. 11	Dec. 26	1	22	9 55 28'5*
Do. <i>f</i> end	Sept. 21	1916 Feb. 5	1	21	9 55 31'3*
White spot each end of S. Tropical Disturbance	Sept. 29	Jan. 16	2	13	9 55 29'7
Dark spots, S. edge S. Equatorial Belt	Oct. 19	Jan. 18	10	6	9 55 37'0
White spots, central rift S. Equatorial Belt	Sept. 27	Jan. 24	2	11	9 55 39'3
Dark spots, N. edge S. Equatorial Belt	Sept. 21	Jan. 8	27	11	9 50 28'1
White spots, N. edge S. Equatorial Belt	Oct. 2	Jan. 14	19	7	9 50 27'5
White spots, S. edge N. Equatorial Belt	Sept. 20	Jan. 18	20	8	9 50 25'0
Dark Spots, S. edge N. Equatorial Belt	Sept. 23	Jan. 4	26	10	9 50 24'6
Dark spots, N. edge N. Equatorial Belt	Sept. 21	Jan. 2	20	12	9 55 27'3
White spots, N. Tropical Zone	Sept. 30	Jan. 18	15	12	9 55 27'6
Dark spots and streaks, N. Temperate Band	Oct. 3	Jan. 5	14	7	9 55 28'0
Dark spots and streaks, N.N. Temperate Band	Oct. 8	Jan. 12	18	7	9 55 34'2
White spots, N.N. Temperate Zone	Oct. 18	Jan. 10	10	4	9 55 38'9
Dark spots and streaks, N. Polar Band and region farther N.	Oct. 21	Jan. 2	15	6	9 55 45'9

Notes.—The Red Spot Hollow exhibited a progressive diminution in length from 37° in September 1915 to about 34° in February 1916. The spot chart indicates that this was due to a quickened rate of velocity of the *f* Shoulder. The Red Spot

* The mean period for both ends of the S. Tropical Disturbance was 9^h 55^m 29'9".

itself, which was easily discernible in the 26-in. reflector, was, as in former years, displaced slightly toward the *f* Shoulder, and the accelerated rate of velocity of the *f* Shoulder appeared finally to render this shoulder in contact with the spot.

A number of white spots in the central rift of the S. Equatorial Belt exhibited a rotation period not dissimilar to that of the Red Spot. This remark applies equally to the white spots of the S. Tropical Zone, except in the instance of those following the S. Tropical Disturbance, where a motion more in conformity with that of the Disturbance was manifested.

The spots along the S. edge of the N. Equatorial Belt exhibited a marked acceleration of velocity, while the corresponding ones along the N. edge of the S. Equatorial Belt moved at the normal rate of $9^h 50^m 27.8^s$. The difference between the two currents works out at 2.9 seconds.

The spot charts show the inconsistency of the rates of motion of the individual spots in nearly all latitudes, the longitudinal drift even of the Red Spot never for any considerable period remaining constant.

The Illumination of the Field of a Photographic Objective
(Supplemental Note). By H. C. Lord.

In a paper published in the *Monthly Notices*, vol. lxxvi. p. 197, there were developed certain formulæ which give, for points in the field at an angular distance ϕ from the centre, the intensity of illumination, I , in terms of that at the centre taken as unity. In the development of these formulæ it was tacitly assumed that there was no interference with the emergent cone of rays by the back combination of the objective. In other words, it was assumed that all the light came out of the objective that was admitted by the front combination. Such, however, is only the case for a comparatively small field, but the formulæ already developed can be readily adapted to include this additional loss.

For central refraction, $\phi=0$, it is evident that the radius, r , of the illuminated circle formed on a plane tangent to the vertex of the last lens of the combination by the emergent cone of rays, is given by

$$r = \frac{u_2 u_3 u_4 \dots u_n}{v_1 v_2 v_3 \dots v_n - 1} \left(\frac{F}{2N} \right)$$

where u denotes the distance from any element to the point where the rays from the preceding element come to focus, and v denotes the distance of this same point from the preceding element. The term "element" is here used as defined by Taylor in his *System of Applied Optics*. In case the constants of the system are not known, the value of r can obviously be

determined experimentally. If D'_n be the distance of the emergent pupil point from the back vertex of the objective, and $2R_n$ be the aperture of the last lens, the value of ϕ for which the back combination just begins to interfere with the emergent cone of rays will be given by

$$\tan \phi_0 = \frac{R_n - r}{D'_n}.$$

So long as ϕ is less than the above value of ϕ_0 no account of the effect of the rear combination need be taken. For values larger than this the formulæ developed in the paper above referred to should be applied as follows. Compute the intensity of the light transmitted by the front combination exactly as if there were no back combination, but omit the factor $\cos \phi$ and call the result I'_1 . Then compute the intensity due to the effect of the back combination alone by the formulæ of the second case considered in the preceding paper, omitting the factor $\cos \phi$, replacing $\frac{F}{2N}$ by r as computed above, and replacing D' by D'_n , and call the result I'_2 . The intensity of the image given by the complete objective will then be given by

$$I = (I'_1 + I'_2 - 1) \cos \phi.$$

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